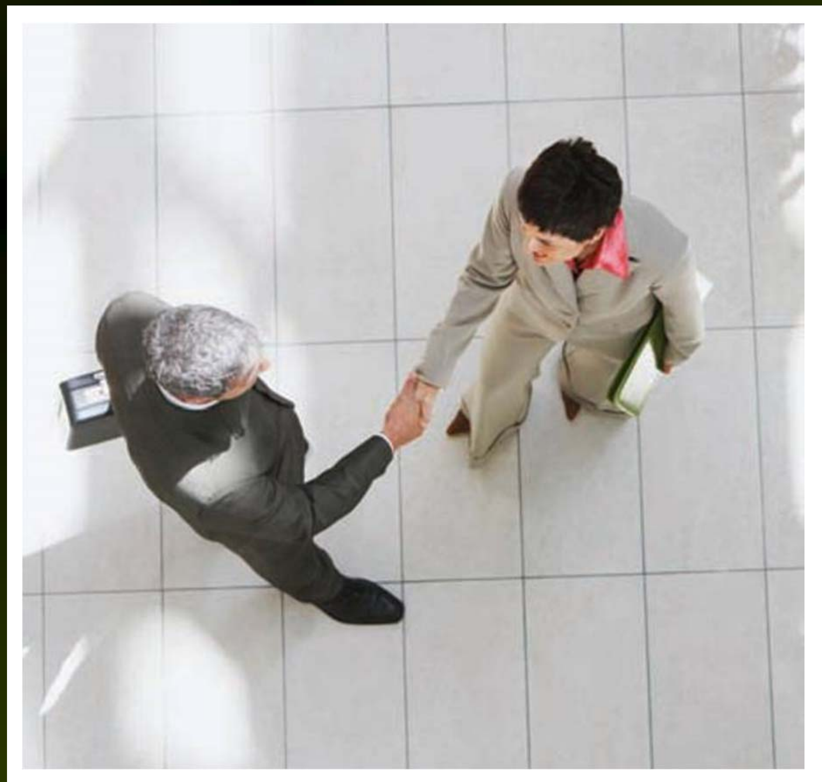
1

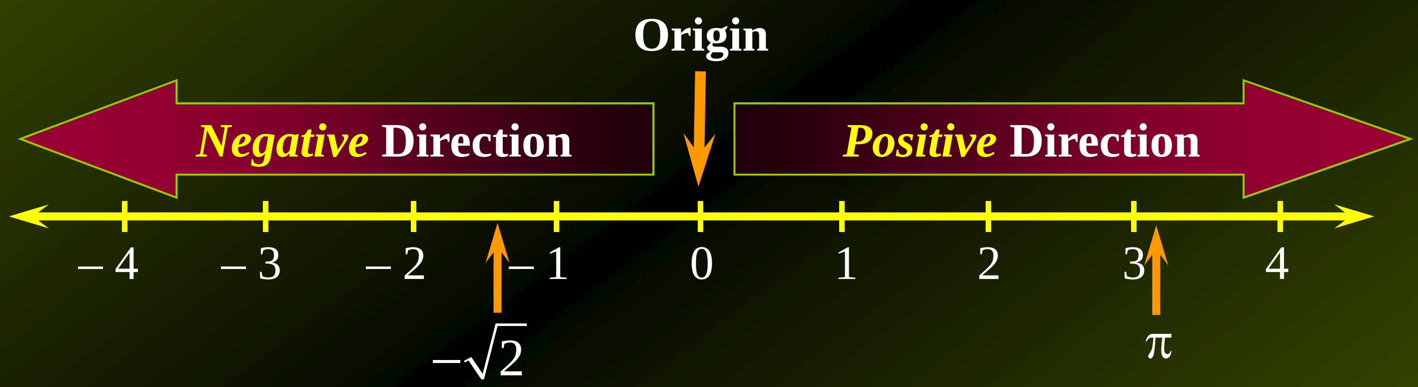
PRELIMINARIES



1.1 Precalculus Review I

The Real Number Line

We can represent real numbers geometrically by points on a real number, or coordinate, line:



This line includes *all* real numbers.

Exactly one point on the line is associated with each real number, and vice-versa.

Finite Intervals

Open Intervals

The set of real numbers that lie *strictly* between two fixed numbers *a* and *b* is called an open interval (*a*, *b*).

It consists of all the real numbers that satisfy the inequalities $a < x < b$.

It is called "open" because neither of its endpoints is included in the interval.

Finite Intervals

Closed Intervals

The set of real numbers that lie between two fixed numbers a and b, that includes a and b, is called a closed interval $[a, b]$.

It consists of all the real numbers that satisfy the inequalities $a \leq x \leq b$.

It is called "closed" because both of its endpoints are included in the interval.

Finite Intervals

Half-Open Intervals

The set of real numbers that between two fixed numbers *a* and *b*, that contains *only one* of its endpoints *a* or *b*, is called a half-open interval $(a, b]$ or $[a, b)$.

It consists of all the real numbers that satisfy the inequalities $a < x \leq b$ or $a \leq x < b$.

Infinite Intervals

Examples of infinite intervals include:

(a, ∞), $[a, \infty)$, $(-\infty, a)$, and $(-\infty, a]$.

The above are defined, respectively, by the set of real numbers that satisfy $x > a$, $x \geq a$, $x < a$, $x \leq a$.

Exponents and Radicals

If b is any real number and n is a positive integer, then the expression b^n is defined as the number

$$b^n = \underbrace{b \cdot b\ b \cdot \cdot \ldots \cdot b}_{n \text{ factors}}$$

The number b is called the base, and the superscript n is called the power of the exponential expression b^n.

For example:

$$2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$$

$$\left(\frac{2}{3}\right)^3 = \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) \cdot \left(\frac{2}{3}\right) = \frac{8}{27}$$

Exponents and Radicals

If $b \neq 0$, we define $b^0 = 1$.

For example:

$2^0 = 1$ and $(-\pi)^0 = 1$, but 0^0 is undefined.

If n is a positive integer, then the expression $b^{1/n}$ is defined to be the number that, when raised to the n^{th} power, is equal to b, thus

$$(b^{1/n})^n = b$$

Such a number is called the n^{th} root of b, also written as

$$\sqrt[n]{b}$$

Exponents and Radicals

Similarly, the expression $b^{p/q}$ is defined as the number

$$(b^{1/q})^p \quad \text{or} \quad \sqrt[q]{b^p}$$

Examples:

$$2^{3/2} = \left(2^{1/2}\right)^3 \approx (1.41412)^3 \approx 2.8283$$

$$4^{-5/2} = \frac{1}{4^{5/2}} = \frac{1}{\left(4^{1/2}\right)^5} = \frac{1}{2^5} = \frac{1}{32}$$

Laws of Exponents

Law	Example
1. $a^m \cdot a^n = a^{m+n}$	$x^2 \cdot x^3 = x^{2+3} = x^5$
2. $\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$	$\frac{x^7}{x^4} = x^{7-4} = x^3$
3. $(a^m)^n = a^{m \cdot n}$	$(x^4)^3 = x^{4 \cdot 3} = x^{12}$
4. $(ab)^n = a^n \cdot b^n$	$(2x)^4 = 2^4 \cdot x^4 = 16x^4$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{x}{2}\right)^3 = \frac{x^3}{2^3} = \frac{x^3}{8}$

Example 1

Simplify the expressions

a. $\left(3x^2\right)\left(4x^3\right) = 12x^{2+3} = 12x^5$

b. $\dfrac{16^{5/4}}{16^{1/2}} = 16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8$

c. $\left(6^{2/3}\right)^3 = 6^{(2/3)(3)} = 6^{6/3} = 6^2 = 36$

d. $\left(x^3y^{-2}\right)^{-2} = \left(x^3\right)^{-2}\left(y^{-2}\right)^{-2} = x^{(3)(-2)}y^{(-2)(-2)} = x^{-6}y^4 = \dfrac{y^4}{x^6}$

e. $\left(\dfrac{y^{3/2}}{x^{1/4}}\right)^{-2} = \dfrac{y^{(3/2)(-2)}}{x^{(1/4)(-2)}} = \dfrac{y^{-3}}{x^{-1/2}} = \dfrac{x^{1/2}}{y^3}$

Example 2

Simplify the expressions (assume *x*, *y*, *m*, and *n* are positive)

a. $\sqrt[4]{16x^4y^8} = \left(16x^4y^8\right)^{1/4} = 16^{1/4}x^{4/4}y^{8/4} = 2xy^2$

b. $\sqrt{12m^3n}\cdot\sqrt{3m^5n} = \sqrt{36m^8n^2} = \left(36m^8n^2\right)^{1/2} = 36^{1/2}m^{8/2}n^{2/2} = 6m^4n$

c. $\dfrac{\sqrt[3]{-27x^6}}{\sqrt[3]{8y^3}} = \dfrac{\left(-27x^6\right)^{1/3}}{\left(8y^3\right)^{1/3}} = \dfrac{-27^{1/3}x^{6/3}}{8^{1/3}y^{3/3}} = -\dfrac{3x^2}{2y}$

Example 3

Rationalize the denominator of the expression

$$\frac{3x}{2\sqrt{x}} = \frac{3x}{2\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{3x\sqrt{x}}{2\sqrt{x^2}}$$

$$= \frac{3x\sqrt{x}}{2x}$$

$$= \frac{3}{2}\sqrt{x}$$

Example 4

Rationalize the numerator of the expression

$$\frac{3\sqrt{x}}{2x} = \frac{3\sqrt{x}}{2x} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$= \frac{3\sqrt{x^2}}{2x\sqrt{x}}$$

$$= \frac{3x}{2x\sqrt{x}}$$

$$= \frac{3}{2\sqrt{x}}$$

Operations With Algebraic Expressions

An algebraic expression of the form ax^n, where the coefficient a is a real number and n is a nonnegative integer, is called a monomial, meaning it consists of one term.

Examples:

$7x^2$ $\quad$ $2xy$ $\quad$ $12x^3y^4$

A polynomial is a monomial or the sum of two or more monomials.

Examples:

$x^2 + 4x + 4$ $\quad$ $x^4 + 3x^2 - 3$ $\quad$ $x^2y - xy + y$

Operations With Algebraic Expressions

Constant terms, or terms containing the same variable factors are called like, or similar, terms.

Like terms may be combined by adding or subtracting their numerical coefficients.

Examples:

$3x + 7x = 10x$ $\qquad$ $12xy - 7xy = 5xy$

Example 5(a)

Simplify the expression

$$(2x^4 + 3x^3 + 4x + 6) - (3x^4 + 9x^3 + 3x^2)$$

$$= 2x^4 + 3x^3 + 4x + 6 - 3x^4 - 9x^3 - 3x^2$$

$$= 2x^4 - 3x^4 + 3x^3 - 9x^3 - 3x^2 + 4x + 6$$

$$= -x^4 - 6x^3 - 3x^2 + 4x + 6$$

Example 5(b)

Simplify the expression

$$2t^3 - \{t^2 - [t - (2t - 1)] + 4\}$$

$$= 2t^3 - \{t^2 - [t - 2t + 1] + 4\}$$

$$= 2t^3 - \{t^2 - [-t + 1] + 4\}$$

$$= 2t^3 - \{t^2 + t - 1 + 4\}$$

$$= 2t^3 - t^2 - t + 1 - 4$$

$$= 2t^3 - t^2 - t - 3$$

Example 6(a)

Perform the operation and simplify the expression

$$(x^2+1)(3x^2+10x+3)$$

$$= x^2(3x^2+10x+3)+1(3x^2+10x+3)$$

$$= 3x^4+10x^3+3x^2+3x^2+10x+3$$

$$= 3x^4+10x^3+6x^2+10x+3$$

Example 6(b)

Perform the operation and simplify the expression

$$(e^t + e^{-t})e^t - e^t(e^t - e^{-t}) = e^{2t} + e^0 - e^{2t} + e^0$$

$$= 1 + 1$$

$$= 2$$

Factoring

Factoring is the process of expressing an algebraic expression as a product of other algebraic expressions.

Example:

$$3x^2 - x = x(3x - 1)$$

Factoring

To factor an algebraic expression, first check to see if it contains any common terms.

If so, factor out the greatest common term.

For example, the greatest common factor for the expression

$$2a^2x + 4ax + 6a$$

is $2a$, because

$$\begin{aligned} 2a^2x + 4ax + 6a &= 2a \cdot ax + 2a \cdot 2x + 2a \cdot 3 \\ &= 2a(ax + 2x + 3) \end{aligned}$$

Example 7

Factor out the greatest common factor in each expression

a. $-0.3t^2 + 3t = -0.3t(t - 10)$

b. $2x^{3/2} - 3x^{1/2} = x^{1/2}(2x - 3)$

c. $2ye^{xy^2} + 2xy^3e^{xy^2} = 2ye^{xy^2}(1 + xy^2)$

Example 8

Factor out the greatest common factor in each expression

a. $2ax + 2ay + bx + by \;= 2a(x + y) + b(x + y)$

$$= (2a + b)(x + y)$$

b. $3x\sqrt{y} - 4 - 2\sqrt{y} + 6x \;= 3x\sqrt{y} - 2\sqrt{y} + 6x - 4$

$$= \sqrt{y}(3x - 2) + 2(3x - 2)$$

$$= (3x - 2)(\sqrt{y} + 2)$$

Factoring Second Degree Polynomials

The factors of the second-degree polynomial with integral coefficients

$$px^2 + qx + r$$

are $(ax + b)(cx + d)$, where $ac = p$, $ad + bc = q$, and $bd = r$.

Since only a limited number of choices are possible, we use a trial-and-error method to factor polynomials having this form.

Example 9(a)

Find the correct factorization for $x^2 - 2x - 3$

Solution:

The x^2 coefficient is 1, so the only possible first degree terms are

$$(x \quad)(x \quad)$$

The product of the constant term is -3, which gives us the following possible factors

$$(x - 1)(x + 3)$$

$$(x + 1)(x - 3)$$

Example 9(a) – *Solution*

cont'd

We check to see which set of factors yields –2 for the x coefficient:

$$(-1)(1) + (1)(3) = 2 \qquad \text{or} \qquad (1)(1) + (1)(-3) = -2$$

and conclude that the correct factorization is

$$x^2 - 2x - 3 = (x + 1)(x - 3)$$

Example 9(b)

Find the correct factorization for the expressions

$$3x^2 + 4x - 4 = (3x-2)(x+2)$$

$$3x^2 - 6x - 24 = 3(x^2 - 2x - 8)$$

$$= 3(x-4)(x+2)$$

Roots of Polynomial Expressions

A polynomial equation of degree n in the variable x is an equation of the form

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

where n is a nonnegative integer and $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$.

For example, the equation

$$-2x^5 + 8x^3 - 6x^2 + 3x + 1 = 0$$

is a polynomial equation of degree 5.

Roots of Polynomial Expressions

The roots of a polynomial equation are the values of *x* that satisfy the equation.

One way to factor the roots of a polynomial equation is to factor the polynomial and then solve the equation.

For example, the polynomial equation

$$x^3 - 3x^2 + 2x = 0$$

may be written in the form

$$x(x^2 - 3x + 2) = 0 \quad \textbf{or} \quad x(x-1)(x-2) = 0$$

Roots of Polynomial Expressions

For the product to be zero, at least one of the factors must be zero, therefore, we have

$$x = 0 \qquad x - 1 = 0 \qquad x - 2 = 0$$

So, the roots of the equation are $x = 0$, 1, and 2.

The Quadratic Formula

The solutions of the equation

$$ax^2 + bx + c = 0 \qquad (a \neq 0)$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 10(a)

Solve the equation using the quadratic formula:

$$2x^2 + 5x - 12 = 0$$

Solution:

For this equation, $a = 2$, $b = 5$, and $c = -12$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-5 \pm \sqrt{5^2 - 4(2)(-12)}}{2(2)}$$

$$= \frac{-5 \pm \sqrt{121}}{4} = \frac{-5 \pm 11}{4}$$

$$= -4 \quad \textbf{or} \quad \frac{3}{2}$$

Example 10(b)

Solve the equation using the quadratic formula:

$$x^2 = -3x + 8$$

Solution

First, rewrite the equation in the standard form

$$x^2 + 3x - 8 = 0$$

For this equation, $a = 1$, $b = 3$, and $c = -8$, so

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{3^2 - 4(1)(-8)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{41}}{2}$$

Example 10(b) – *Solution*

cont'd

So,

$$x \approx \frac{-3+\sqrt{41}}{2} = 1.7$$

or

$$x \approx \frac{-3-\sqrt{41}}{2} = -4.7$$