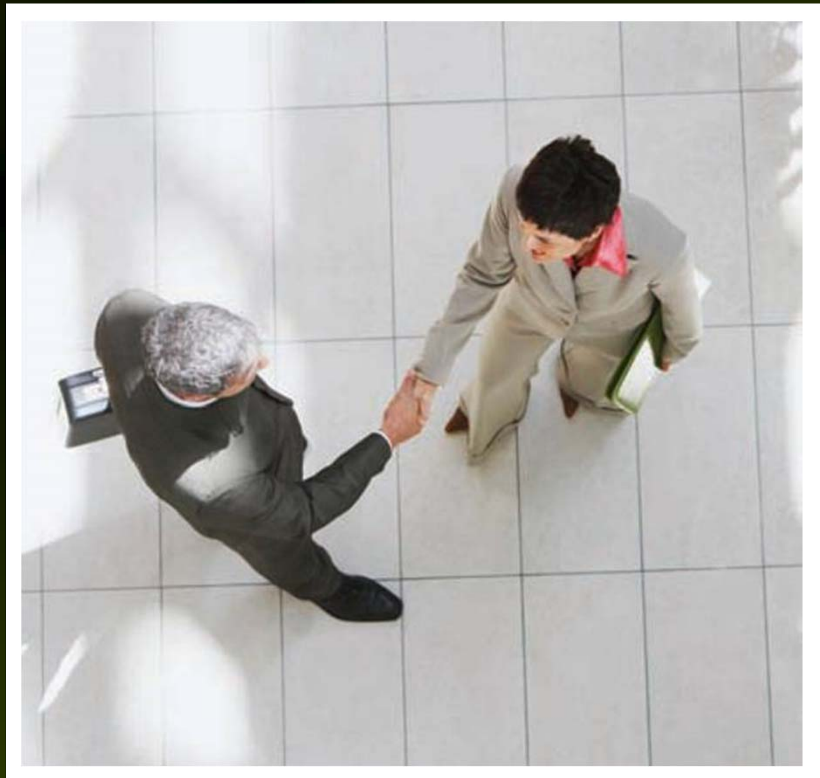
1 PRELIMINARIES



1.2 Precalculus Review II

Rational Expressions

Quotients of polynomials are called rational expressions.

For example

$$\frac{-3x+8}{2x+3} \qquad \frac{5x^2y^3-2xy}{4x} \qquad \frac{2}{5ab}$$

Rational Expressions

The properties of real numbers apply to rational expressions.

Examples

Using the properties of number we may write

$$\frac{ac}{bc} = \frac{a}{b} \cdot \frac{c}{c} = \frac{a}{b} \cdot 1 = \frac{a}{b}$$

where a, b, and c are any real numbers and b and c are not zero.

Similarly, we may write

$$\frac{(x+2)(x-3)}{(x-2)(x-3)} = \frac{(x+2)}{(x-2)} \qquad (x \neq 2, 3)$$

Example 1(a)

Simplify the expression

$$\frac{x^2+2x-3}{x^2+4x+3}=\frac{(x+3)(x-1)}{(x+3)(x+1)}$$

$$=\frac{x-1}{x+1}$$

Example 1(b)

Simplify the expression

$$\frac{(x^2+1)^2(-2)+(2x)(2)(x^2+1)(2x)}{(x^2+1)^4} = \frac{(x^2+1)\left[(x^2+1)(-2)+(2x)(2)(2x)\right]}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(-2x^2-2+8x^2)}{(x^2+1)^4}$$

$$= \frac{(x^2+1)(6x^2-2)}{(x^2+1)^4}$$

$$= \frac{(6x^2-2)}{(x^2+1)^3}$$

$$= \frac{2(3x^2-1)}{(x^2+1)^3}$$

Rules of Multiplication and Division

If P, Q, R, and S are polynomials, then

Multiplication

$$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS} \qquad (Q, S \neq 0)$$

Division

$$\frac{P}{Q}\div\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}=\frac{PS}{QR} \qquad (Q, R, S \neq 0)$$

Example 2

Perform the indicated operation and simplify

$$\frac{2x-8}{x+2}\cdot\frac{x^2+4x+4}{x^2-16}=\frac{2(x-4)}{x+2}\cdot\frac{(x+2)^2}{(x+4)(x-4)}$$

$$=\frac{2(x-4)(x+2)(x+2)}{(x+2)(x+4)(x-4)}$$

$$=\frac{2(x+2)}{x+4}$$

Rules of Addition and Subtraction

If P, Q, R, and S are polynomials, then

Addition

$$\frac{P}{R}+\frac{Q}{S}=\frac{P+Q}{R} \qquad (R \neq 0)$$

Subtraction

$$\frac{P}{R}-\frac{Q}{S}=\frac{P-Q}{R} \qquad (R \neq 0)$$

Example 3(b)

Perform the indicated operation and simplify

$$\frac{1}{x+h}-\frac{1}{x}=\frac{x-(x+h)}{x(x+h)}$$

$$=\frac{x-x-h}{x(x+h)}$$

$$=\frac{-h}{x(x+h)}$$

Other Algebraic Fractions

The techniques used to simplify rational expressions may also be used to simplify algebraic fractions in which the numerator and denominator are not polynomials.

Example 4(a)

Simplify

$$\frac{1+\frac{1}{x+1}}{x-\frac{4}{x}} = \frac{\frac{x+1+1}{x+1}}{\frac{x^2-4}{x}}$$

$$= \frac{x+2}{x+1} \cdot \frac{x}{x^2-4}$$

$$= \frac{x+2}{x+1} \cdot \frac{x}{(x+2)(x-2)}$$

$$= \frac{x}{(x+1)(x-2)}$$

Example 4(b)

Simplify

$$\frac{x^{-1}+y^{-1}}{x^{-2}-y^{-2}} = \frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x^2}-\frac{1}{y^2}} = \frac{\frac{y+x}{xy}}{\frac{y^2-x^2}{x^2y^2}}$$

$$= \frac{y+x}{xy}\cdot\frac{x^2y^2}{y^2-x^2}$$

$$= \frac{y+x}{xy}\cdot\frac{(xy)^2}{(y+x)(y-x)}$$

$$= \frac{xy}{y-x}$$

Rationalizing Algebraic Fractions

When the denominator of an algebraic fraction contains sums or differences involving radicals, we may rationalize the denominator.

To do so we make use of the fact that

$$\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)=\left(\sqrt{a}\right)^2-\left(\sqrt{b}\right)^2$$
$$=a-b$$

Example 6

Rationalize the denominator

$$\frac{1}{1+\sqrt{x}} = \frac{1}{1+\sqrt{x}} \cdot \frac{1-\sqrt{x}}{1-\sqrt{x}}$$

$$= \frac{1-\sqrt{x}}{(1)^2 - \left(\sqrt{x}\right)^2}$$

$$= \frac{1-\sqrt{x}}{1-x}$$

Example 7

Rationalize the numerator

$$\frac{\sqrt{1+h}-1}{h}=\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\frac{\left(\sqrt{1+h}\right)^2-(1)^2}{h\left(\sqrt{1+h}+1\right)}$$

$$=\frac{1+h-1}{h\left(\sqrt{1+h}+1\right)}$$

$$=\frac{h}{h\left(\sqrt{1+h}+1\right)}$$

$$=\frac{1}{\sqrt{1+h}+1}$$

Properties of Inequalities

If a, b, and c, are any real numbers, then

Property 1 If $a < b$ and $b < c$, then $a < c$.

Property 2 If $a < b$, then $a + c < b + c$.

Property 3 If $a < b$ and $c > 0$, then $ac < bc$.

Property 4 If $a < b$ and $c < 0$, then $ac > bc$.

Example 8

Find the set of real numbers that satisfy

$$-1 \leq 2x - 5 < 7$$

Solution:

Add 5 to each member of the given double inequality

$$4 \leq 2x < 12$$

Multiply each member of the inequality by ½

$$2 \leq x < 6$$

So, the solution is the set of all values of x lying in the interval $[2, 6)$.

Example 9

Solve the inequality $x^2 + 2x - 8 < 0$.

Solution:

Factorizing we get $(x + 4)(x - 2) < 0$.

For the product to be negative, the factors must have *opposite signs*, so we have two possibilities to consider:

1. The inequality holds if $(x + 4) < 0$ and $(x - 2) > 0$, which means $x < -4$, and $x > 2$, but *this is impossible*: x cannot meet these two conditions simultaneously.

Example 9 – *Solution*

cont'd

2. The inequality also holds if $(x + 4) > 0$ and $(x – 2) < 0$, which means $x > – 4$, and $x < 2$, or $– 4 < x < 2$.

So, the solution is the set of all values of x lying in the interval $(–4, 2)$.

Example 10

Solve the inequality $\frac{x+1}{x-1} \geq 0$

Solution:

For the quotient to be positive, the numerator and denominator must have the *same sign*, so we have two possibilities to consider:

1. The inequality holds if $(x + 1) \leq 0$ and $(x - 1) < 0$, which means $x \leq -1$, and $x < 1$, both of these conditions are met only when $x \leq -1$.

Example 10 – *Solution*

cont'd

2. The inequality also holds if $(x + 1) \geq 0$ and $(x - 1) > 0$, which means $x \geq -1$, and $x > 1$, both of these conditions are met only when $x > 1$.

So, the solution is the set of all values of x lying in the intervals $(-\infty, -1]$ and $(1, \infty)$.

Absolute Value

The absolute value of a number *a* is denoted |*a*| and is defined by

$$|a| = \begin{cases} a & \textbf{if } a \geq 0 \\ -a & \textbf{if } a < 0 \end{cases}$$

Absolute Value Properties

If *a*, *b*, and *c*, are any real numbers, then

Property 5 $|-a| = |a|$

Property 6 $|ab| = |a|\,|b|$

Property 7 $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|} \quad (b \neq 0)$

Property 8 $|a + b| \leq |a| + |b|$

Example 12(a)

Evaluate the expression

$$|\pi - 5| + 3$$

Solution:

Since $\pi - 5 < 0$, we see that

$$|\pi - 5| = -(\pi - 5).$$

Therefore

$$\begin{aligned} |\pi - 5| + 3 &= -(\pi - 5) + 3 \\ &= 8 - \pi \\ &\approx 4.8584 \end{aligned}$$

Example 12(b)

Evaluate the expression

$$\left|\sqrt{3}-2\right|+\left|2-\sqrt{3}\right|$$

Solution:

Since $\sqrt{3}-2<0$, we see that $\left|\sqrt{3}-2\right|=-\left(\sqrt{3}-2\right)$

Similarly, $2-\sqrt{3}>0$, so $\left|2-\sqrt{3}\right|=2-\sqrt{3}$

Therefore,

$$\left|\sqrt{3}-2\right|+\left|2-\sqrt{3}\right|=-\left(\sqrt{3}-2\right)+\left(2-\sqrt{3}\right)$$

$$=4-2\sqrt{3}$$

Example 12(b) – *Solution*

cont'd

$$= 2\left(2 - \sqrt{3}\right)$$

$$\approx 0.5359$$

Example 13

Evaluate the inequality $|x| \leq 5$.

Solution:

If $x \geq 0$, then $|x| = x$, so $|x| \leq 5$ implies $x \leq 5$.

If $x < 0$, then $|x| = -x$, so $|x| \leq 5$ implies $-x \leq 5$ or $x \geq -5$.

So, $|x| \leq 5$ means $-5 \leq x \leq 5$, and the solution is $[-5, 5]$.

Example 14

Evaluate the inequality $|2x - 3| \leq 1$.

Solution:

From our last example, we know that $|2x - 3| \leq 1$ is equivalent to $-1 \leq 2x - 3 \leq 1$.

Adding 3 throughout we get $2 \leq 2x \leq 4$.

Dividing by 2 throughout we get $1 \leq x \leq 2$, so the solution is $[1, 2]$.