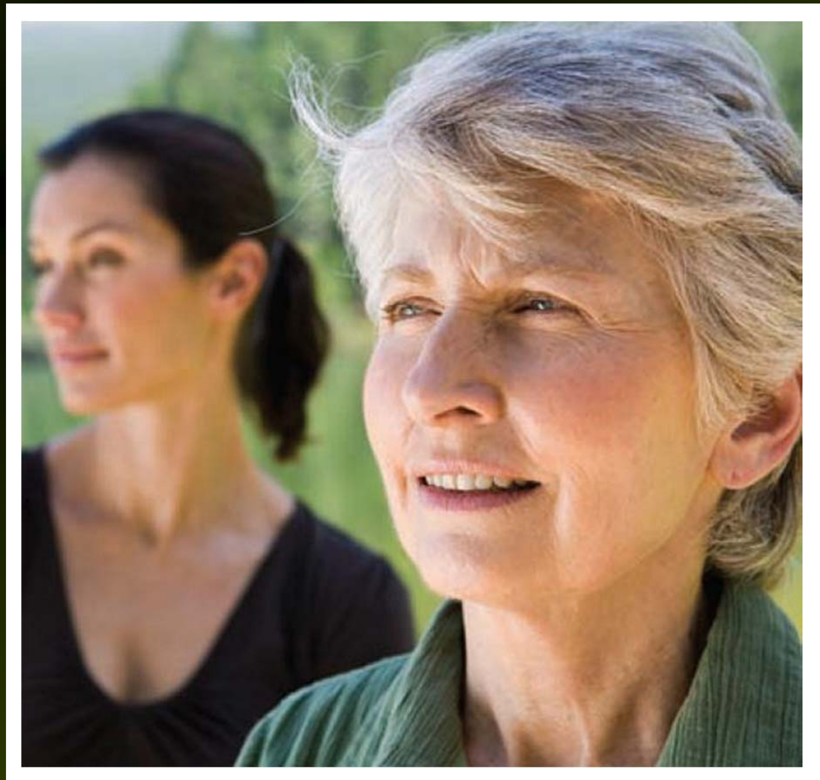
2 FUNCTIONS, LIMITS, AND THE DERIVATIVE



2.1 Functions and Their Graphs

Functions

Function: A function is a rule that assigns to each element in a set *A* one and only one element in a set *B*.

The set *A* is called the domain of the function.

It is customary to denote a function by a letter of the alphabet, such as the letter *f*.

Functions

The element in B that f associates with x is written $f(x)$ and is called the value of f at x.

The set of all the possible values of $f(x)$ resulting from all the possible values of x in its domain, is called the range of $f(x)$.

The output $f(x)$ associated with an input x is unique:

Each x must correspond to one and only one value of $f(x)$.

Example 1(a)

Let the function *f* be defined by the rule

$$f(x) = 2x^2 - x + 1$$

Find: $f(1)$

Solution:

$$f(1) = 2(1)^2 - (1) + 1$$

$$= 2 - 1 + 1 = 2$$

Example 1(b)

Let the function *f* be defined by the rule

$$f(x) = 2x^2 - x + 1$$

Find: $f(-2)$

Solution:

$$f(-2) = 2(-2)^2 - (-2) + 1$$

$$= 8 + 2 + 1 = 11$$

Example 1(c)

Let the function *f* be defined by the rule

$$f(x) = 2x^2 - x + 1$$

Find: $f(a)$

Solution:

$$f(a) = 2(a)^2 - (a) + 1$$

$$= 2a^2 - a + 1$$

Example 1(d)

Let the function *f* be defined by the rule

$$f(x) = 2x^2 - x + 1$$

Find: $f(a + h)$

Solution:

$$f(a+h) = 2(a+h)^2 - (a+h) + 1$$

$$= 2a^2 + 4ah + 2h^2 - a - h + 1$$

Applied Example 2

ThermoMaster manufactures an indoor-outdoor thermometer at its Mexican subsidiary. Management estimates that the profit (in dollars) realizable by ThermoMaster in the manufacture and sale of x thermometers per week is

$$P(x) = -0.001x^2 + 8x - 5000$$

Find ThermoMaster's weekly profit if its level of production is:

a. 1000 thermometers per week.

b. 2000 thermometers per week.

Applied Example 2 – *Solution*

We have

$$P(x) = -0.001x^2 + 8x - 5000$$

a. The weekly profit by producing 1000 thermometers is

$$P(1000) = -0.001(1000)^2 + 8(1000) - 5000 = 2000$$

or $2,000.

b. The weekly profit by producing 2000 thermometers is

$$P(2000) = -0.001(2000)^2 + 8(2000) - 5000 = 7000$$

or $7,000.

Determining the Domain of a Function

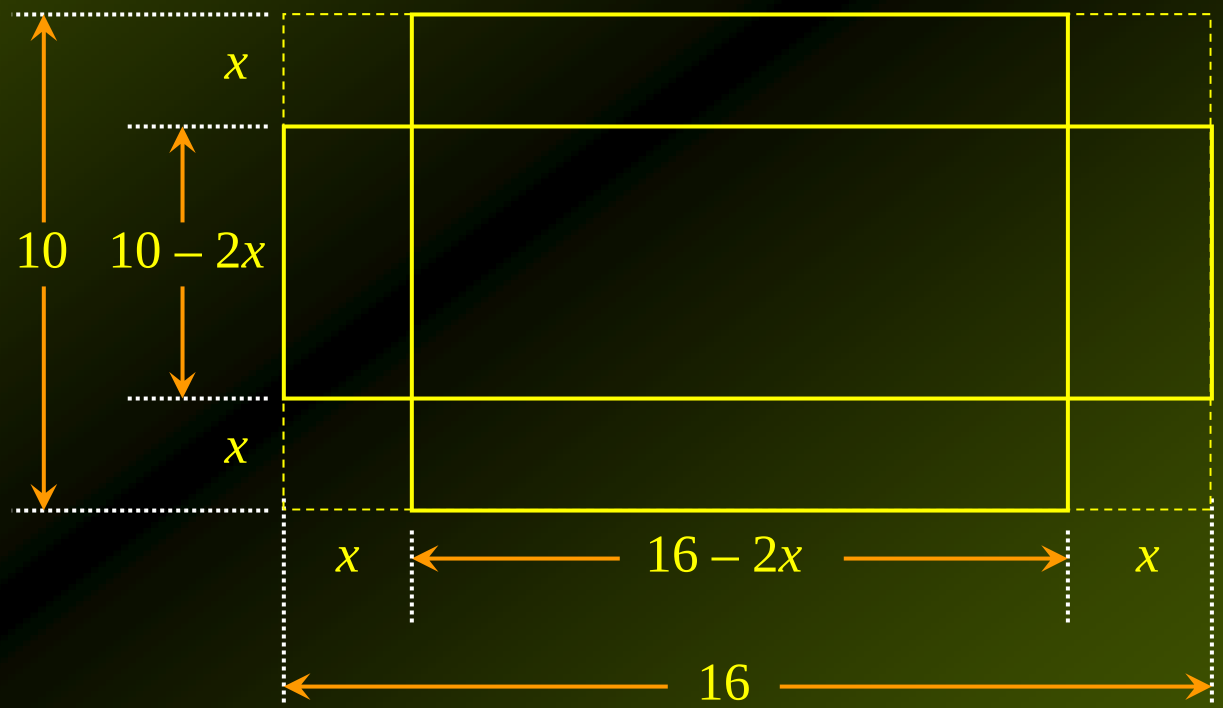
Suppose we are given the function $y = f(x)$. Then, the variable x is called the independent variable. The variable y, whose value depends on x, is called the dependent variable.

To determine the domain of a function, we need to find what restrictions, if any, are to be placed on the independent variable x.

In many practical problems, the domain of a function is dictated by the nature of the problem.

Applied Example 3 – *Packaging*

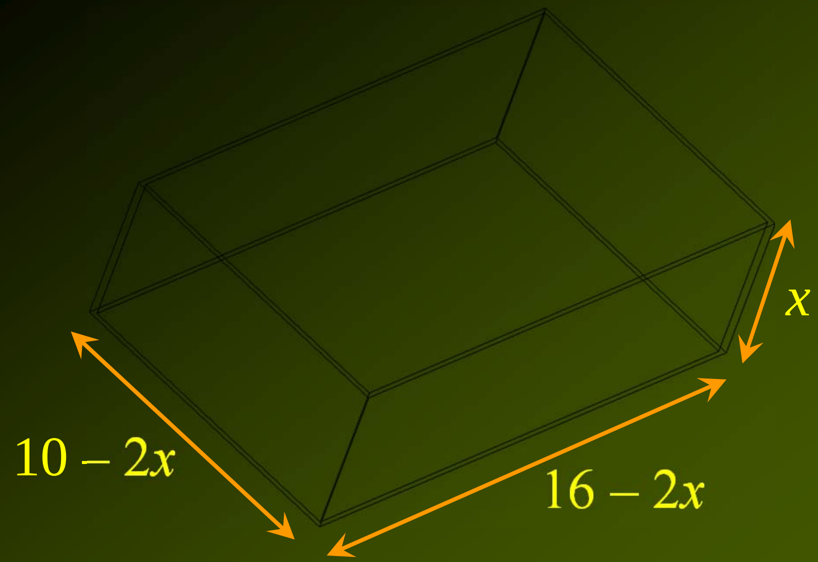
An open box is to be made from a rectangular piece of cardboard 16 inches wide by cutting away identical squares (x inches by x inches) from each corner and folding up the resulting flaps.



Applied Example 3 – *Packaging* cont’d

An open box is to be made from a rectangular piece of cardboard 16 inches wide by cutting away identical squares (*x* inches by *x* inches) from each corner and folding up the resulting flaps.

The dimensions of the resulting box are:



a. Find the expression that gives the volume *V* of the box as a function of *x*.

b. What is the domain of the function?

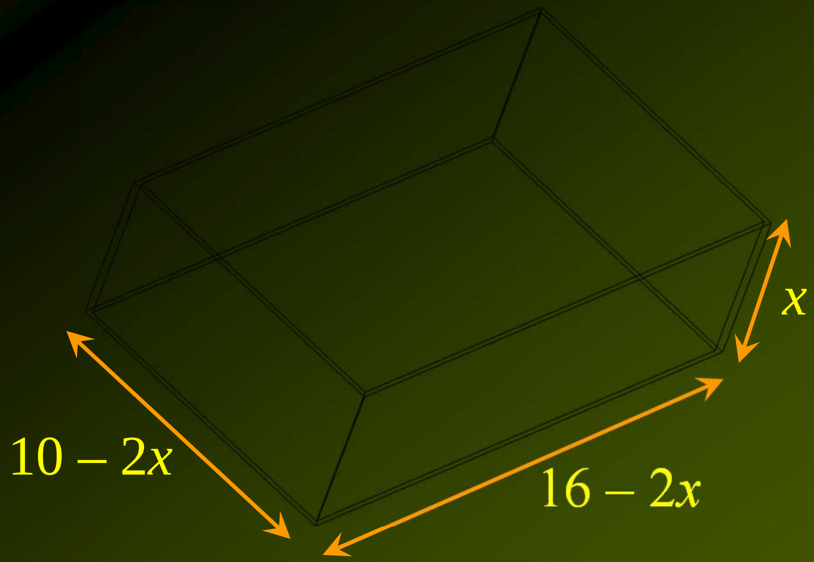
Applied Example 3(a) – *Solution*

The volume of the box is given by multiplying its dimensions (length · width · height), so:

$$V = f(x) = (16 - 2x) \cdot (10 - 2x) \cdot x$$

$$= (160 - 52x + 4x^2)x$$

$$= 4x^3 - 52x^2 + 160x$$



Applied Example 3(b) – *Solution* cont'd

Since the length of each side of the box must be greater than or equal to zero, we see that

$$16 - 2x \geq 0 \qquad 10 - 2x \geq 0 \qquad x \geq 0$$

must be satisfied simultaneously.

Simplified:

$$x \leq 8 \qquad x \leq 5 \qquad x \geq 0$$

All three are satisfied simultaneously provided that:

$$0 \leq x \leq 5$$

Thus, the domain of the function *f* is the interval [0, 5].

Example 4(a)

Find the domain of the function:

$$f(x) = \sqrt{x-1}$$

Solution:

Since the square root of a negative number is undefined, it is necessary that $x - 1 \geq 0$.

Thus the domain of the function is $[1,\infty)$.

Example 4(b)

Find the domain of the function:

$$f(x) = \frac{1}{x^2 - 4}$$

Solution:

Our only constraint is that you cannot divide by zero, so

$$x^2 - 4 \neq 0$$

Which means that

$$x^2 - 4 = (x+2)(x-2) \neq 0$$

Or more specifically $x \neq -2$ and $x \neq 2$.

Thus the domain of f consists of the intervals $(-\infty, -2)$, $(-2, 2)$, $(2, \infty)$.

Example 4(c)

Find the domain of the function:

$$f\left(x\right)=x^2+3$$

Solution:

Here, any real number satisfies the equation, so the domain of *f* is the set of all real numbers.

Graphs of Functions

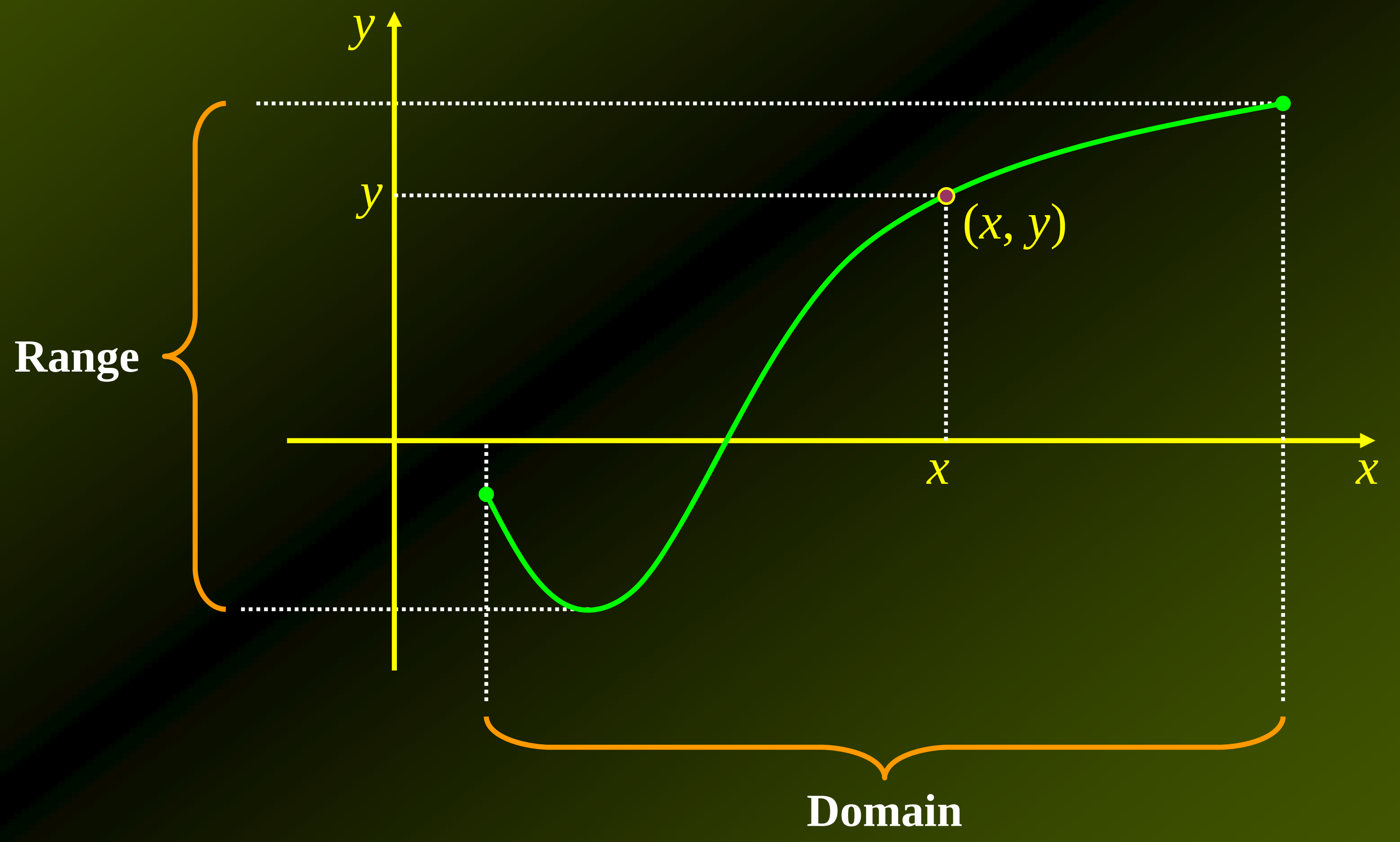
If f is a function with domain A, then corresponding to each real number x in A there is precisely one real number $f(x)$.

Thus, a function f with domain A can also be defined as the set of all ordered pairs $(x, f(x))$ where x belongs to A.

The graph of a function f is the set of all points (x, y) in the xy-plane such that x is the domain of f and $y = f(x)$.

Example 5

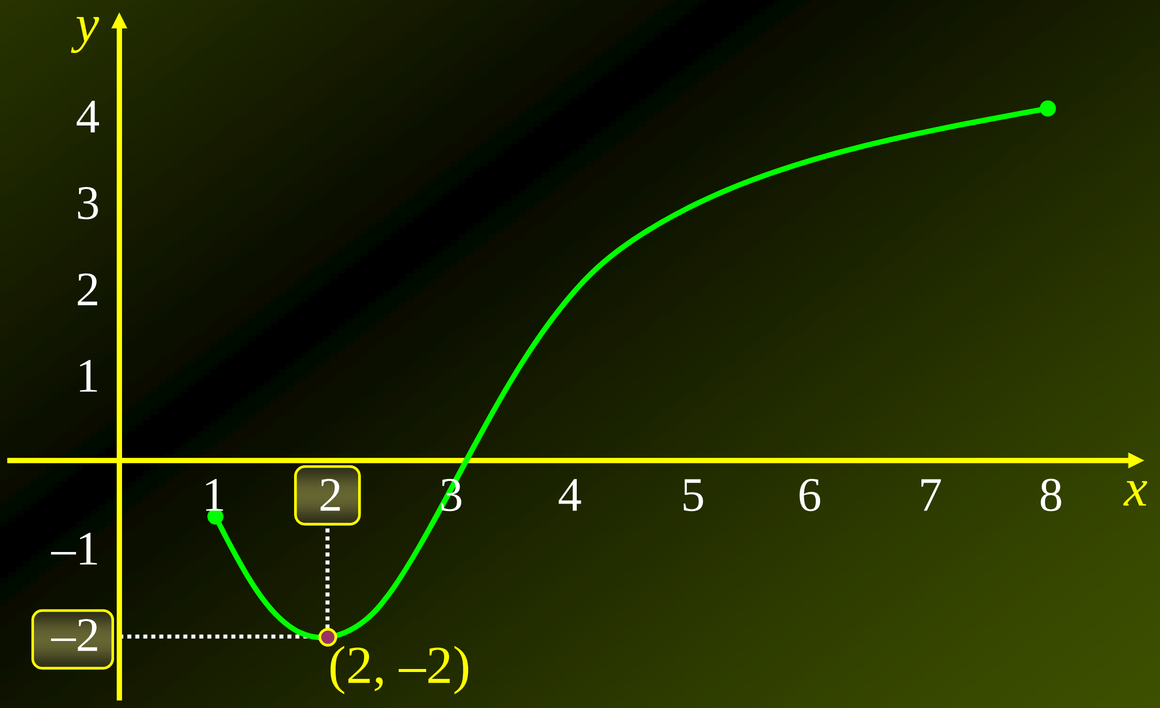
The graph of a function *f* is shown below:



Example 5

a. What is the value of *f*(2)?

Solution:

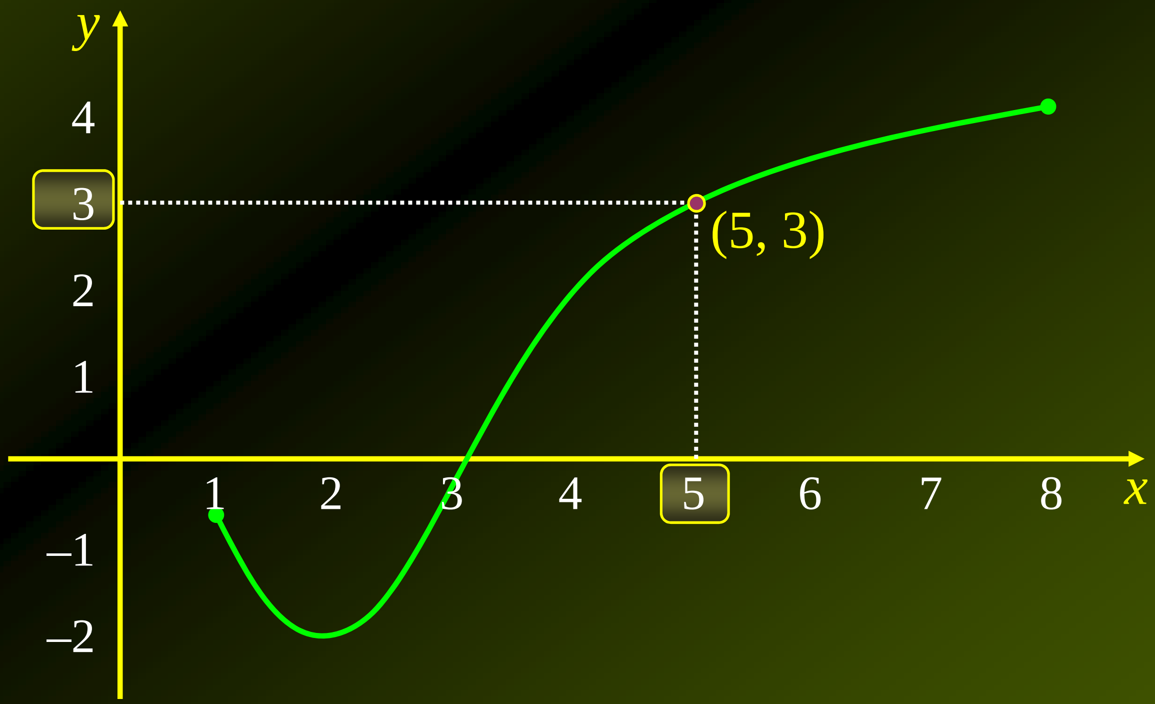


Example 5

cont'd

b. What is the value of $f(5)$?

Solution:

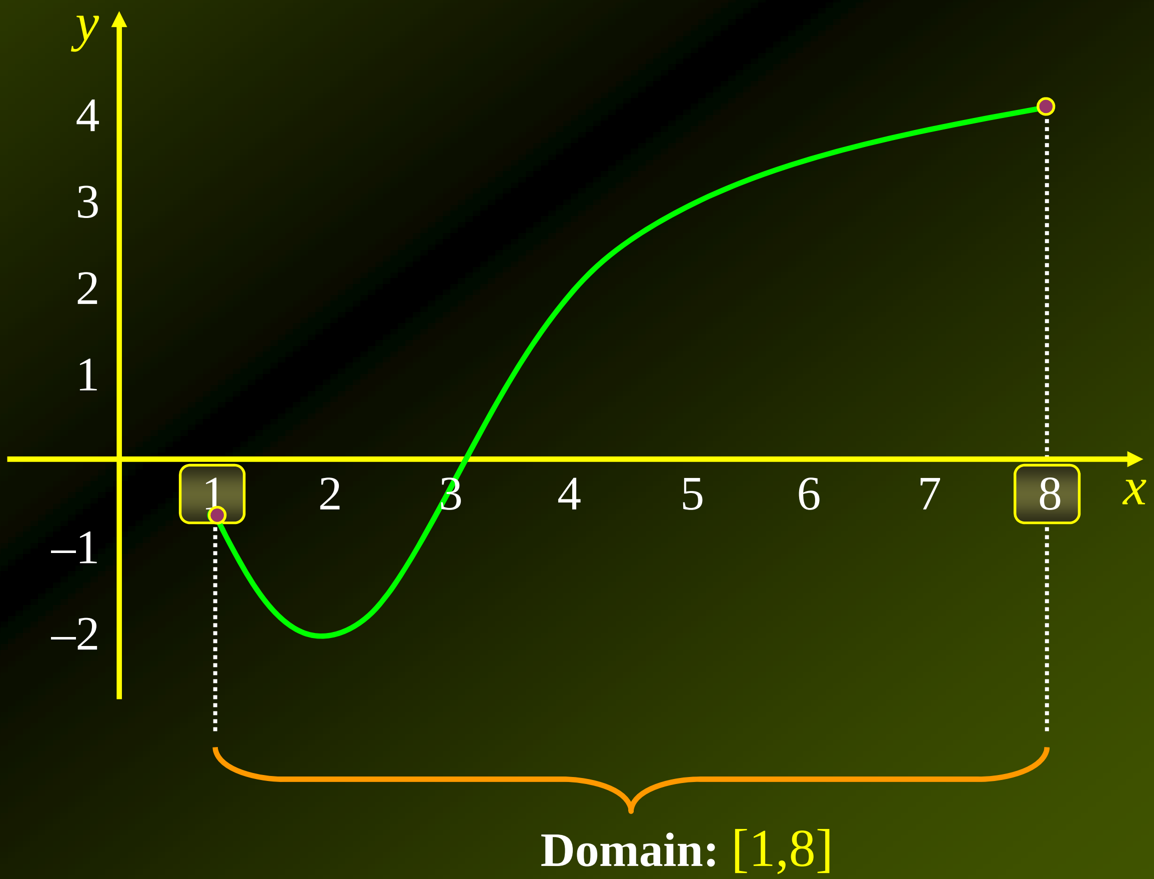


Example 5

cont'd

c. What is the domain of $f(x)$?

Solution:

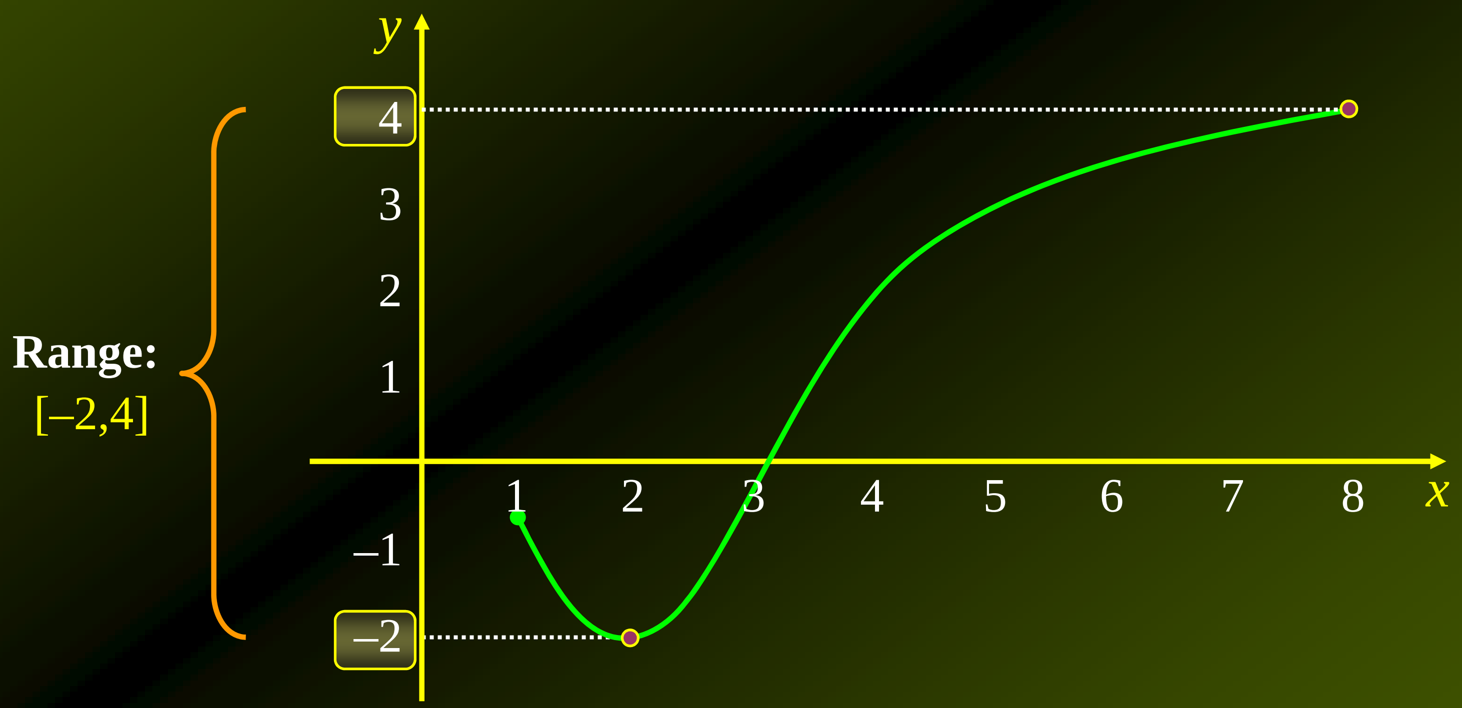


Example 5

cont'd

d. What is the range of $f(x)$?

Solution:



Example 6 – *Sketching a Graph*

Sketch the graph of the function defined by the equation

$$y = x^2 + 1$$

Solution:

The domain of the function is the set of all real numbers.

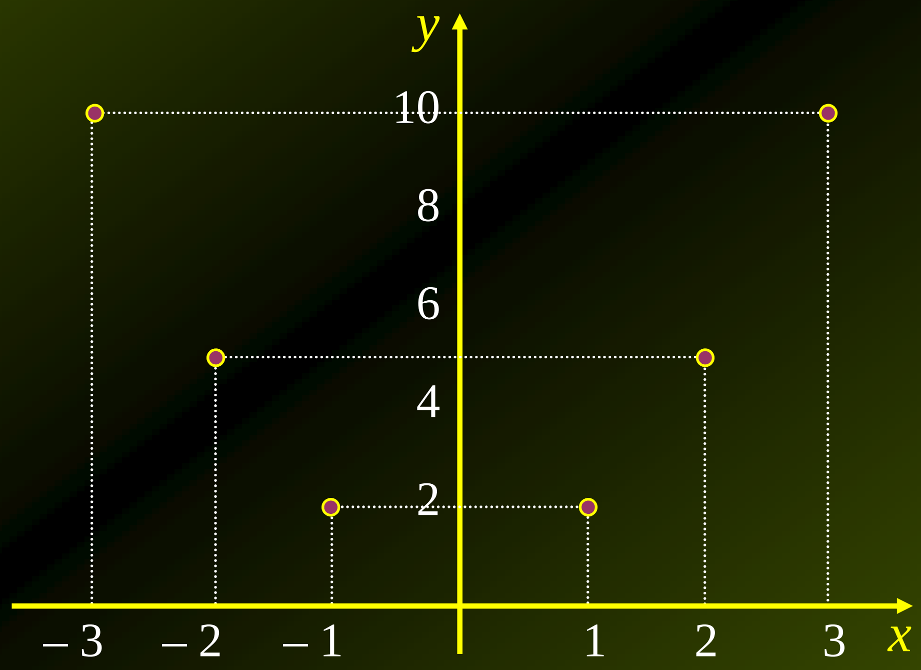
Assign several values to the variable x and compute the corresponding values for y:

x	y
–3	10
–2	5
–1	2
0	1
1	2
2	5
3	10

Example 6 – *Solution*

cont'd

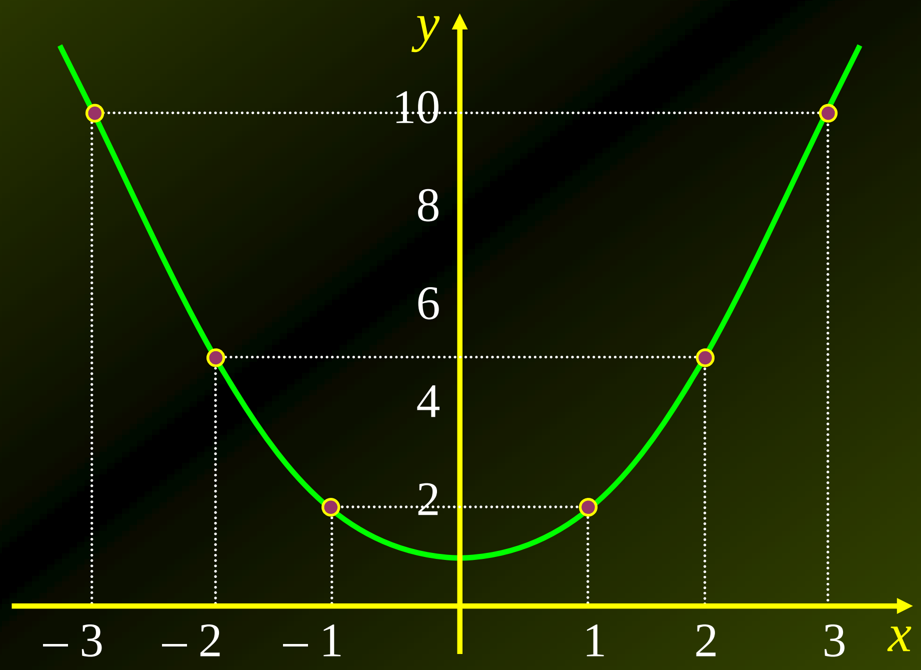
Then plot these values in a graph:



Example 6 – *Solution*

cont'd

And finally, connect the dots:



Example 7 – *Sketching a Graph*

Sketch the graph of the function defined by the equation

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ \sqrt{x} & \text{if } x \geq 0 \end{cases}$$

Solution:

The function *f* is defined in a piecewise fashion on the set of all real numbers.

In the subdomain (–∞, 0), the rule for *f* is given by

$$f(x) = -x$$

In the subdomain [0, ∞), the rule for *f* is given by

$$f(x) = \sqrt{x}$$

Example 7 – *Solution*

cont'd

Substituting negative values for *x* into $f(x) = -x$, while substituting zero and positive values into $f(x) = \sqrt{x}$ we get:

x	y
–3	3
–2	2
–1	1
0	0
1	1
2	1.41
3	1.73

Example 7 – *Solution*

cont'd

Plotting these data and graphing we get: