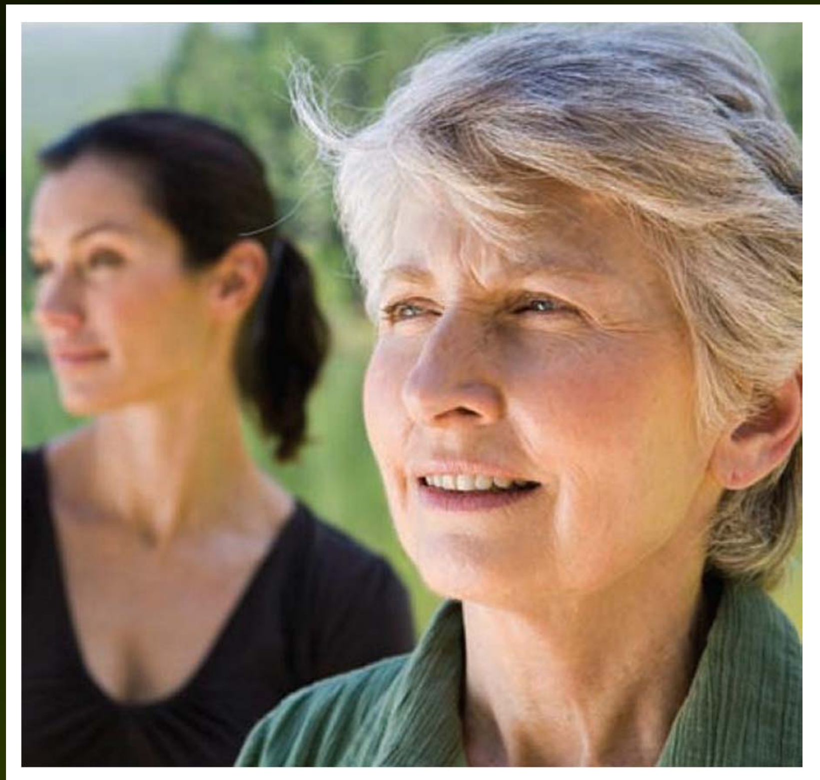
# 2 FUNCTIONS, LIMITS, AND THE DERIVATIVE

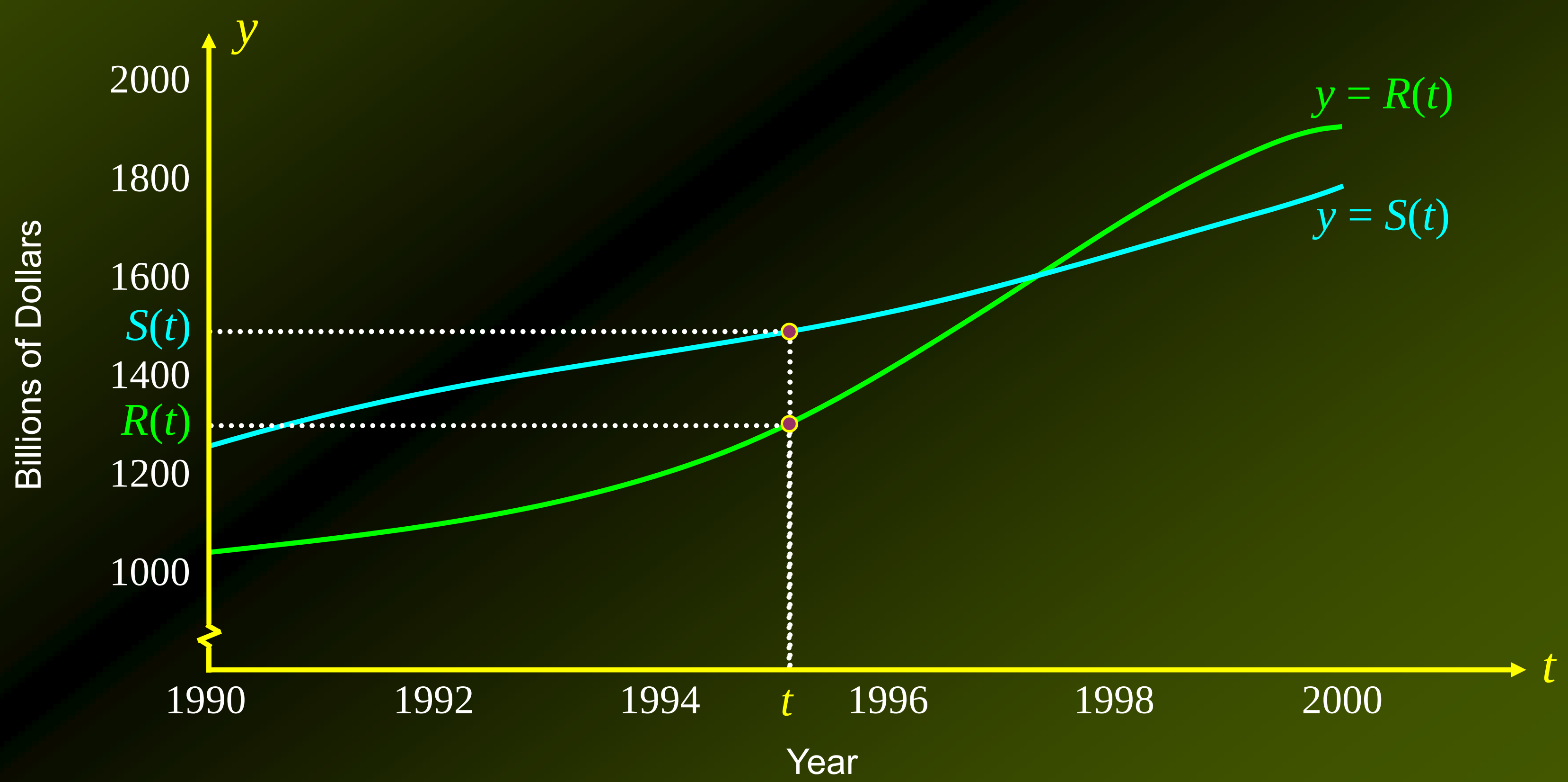


# 2.2 The Algebra of Functions

# The Sum, Difference, Product and Quotient of Functions

Consider the graph below:

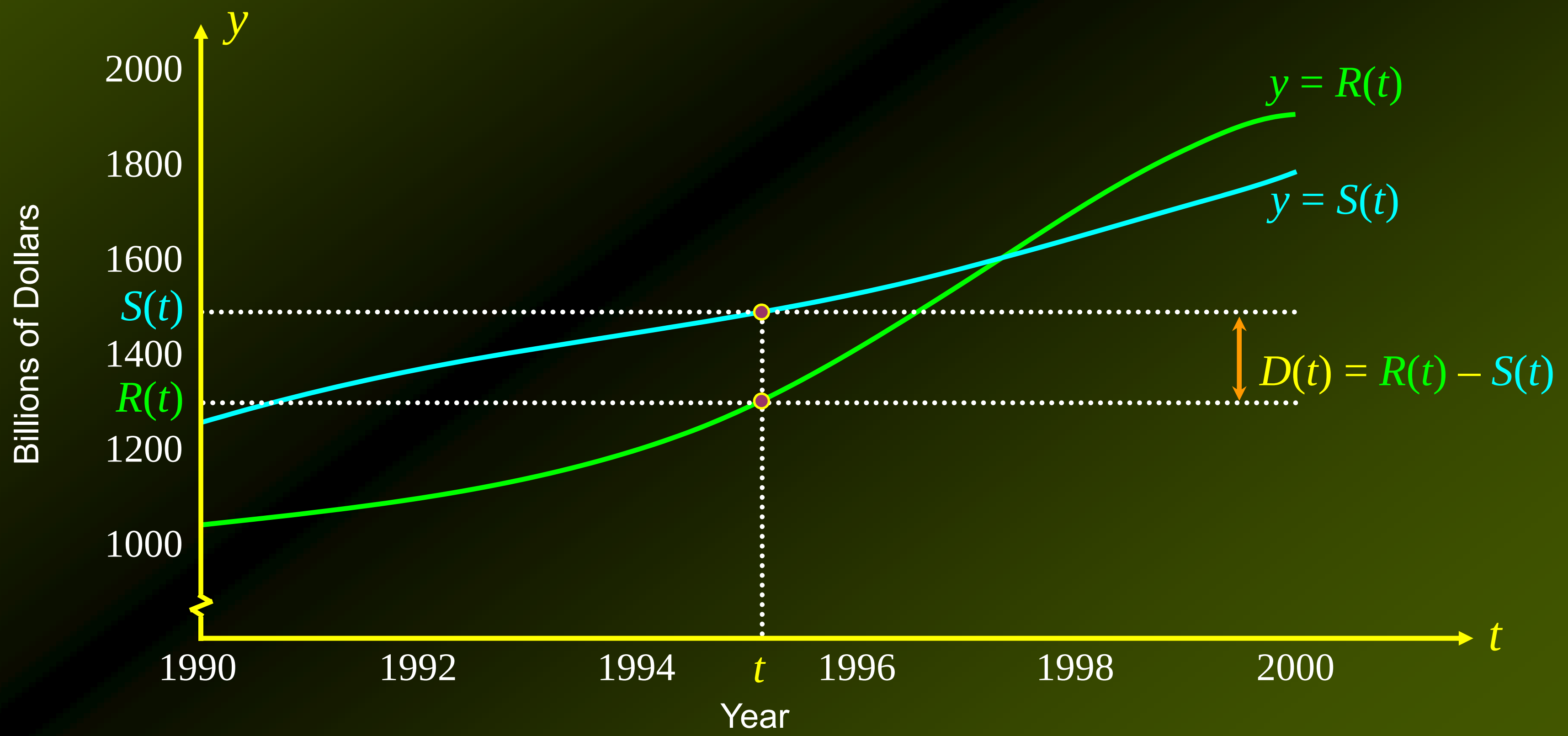
- *R(t)* denotes the federal government revenue at any time *t*.
- *S(t)* denotes the federal government spending at any time *t*.



# The Sum, Difference, Product and Quotient of Functions

Consider the graph below:

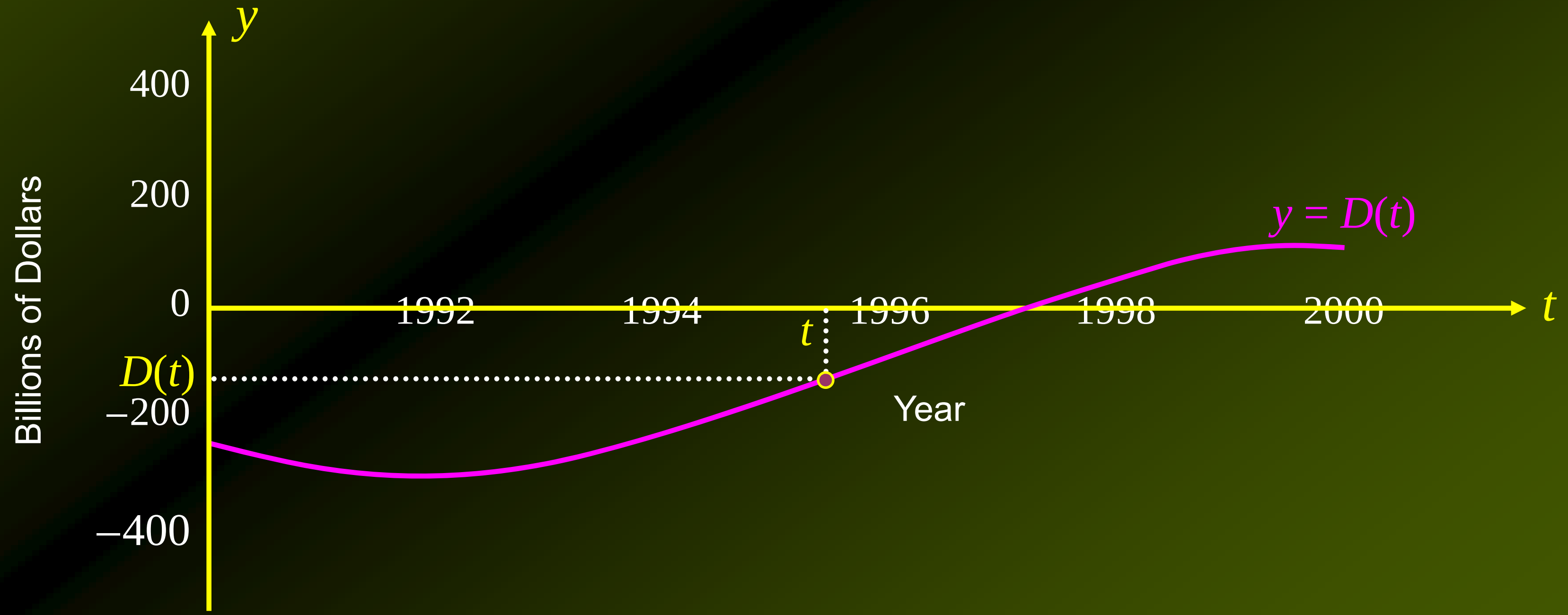
The difference $R(t) – S(t)$ gives the budget deficit (if negative) or surplus (if positive) in billions of dollars at any time $t$.



# The Sum, Difference, Product and Quotient of Functions

The budget balance $D(t)$ is shown below:

- $D(t)$ is also a function that denotes the federal government deficit (surplus) at any time $t$.
- This function is the difference of the two function $R$ and $S$.
- $D(t)$ has the same domain as $R(t)$ and $S(t)$.



# The Sum, Difference, Product and Quotient of Functions

Most functions are built up from other, generally simpler functions.

For example, we may view the function $f(x) = 2x + 4$ as the sum of the two functions $g(x) = 2x$ and $h(x) = 4$.

# The Sum, Difference, Product and Quotient of Functions

Let $f$ and $g$ be functions with domains $A$ and $B$, respectively. The sum $f + g$, the difference $f - g$, and the product $fg$ of $f$ and $g$ are functions with domain $A \cap B$ and rule given by

| | |
|---|---|
| $(f+g)(x) = f(x) + g(x)$ | Sum |
| $(f-g)(x) = f(x) - g(x)$ | Difference |
| $(fg)(x) = f(x)g(x)$ | Product |

The quotient $f/g$ of $f$ and $g$ has domain $A \cap B$ excluding all numbers $x$ such that $g(x) = 0$ and rule given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{Quotient}$$

# Example 1

Let $f(x) = \sqrt{x+1}$ and $g(x) = 2x + 1$.

Find the sum $s$, the difference $d$, the product $p$, and the quotient $q$ of the functions $f$ and $g$.

Solution:

Since the domain of $f$ is $A = [-1,\infty)$ and the domain of $g$ is $B = (-\infty, \infty)$, we see that the domain of $s$, $d$, and $p$ is $A \cap B = [-1,\infty)$.

The rules are as follows:

$$s(x) = (f+g)(x) = f(x) + g(x) = \sqrt{x+1} + 2x + 1$$

$$d(x) = (f-g)(x) = f(x) - g(x) = \sqrt{x+1} - 2x - 1$$

$$p(x) = (fg)(x) = f(x)g(x) = \left(2x+1\right)\sqrt{x+1}$$

# Example 1 – *Solution*

cont'd

The domain of the quotient function is $[-1, \infty)$ together with the restriction $x \neq -½$.

Thus, the domain is $[-1, -½) \cup (-½, \infty)$.

The rule is as follows:

$$q(x) = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x+1}}{2x+1}$$

# Applied Example 2

Suppose Puritron, a manufacturer of water filters, has a monthly fixed cost of $10,000 and a variable cost of

$$-0.0001x^2 + 10x \qquad (0 \leq x \leq 40{,}000)$$

dollars, where $x$ denotes the number of filters manufactured per month. Find a function $C$ that gives the total monthly cost incurred by Puritron in the manufacture of $x$ filters.

# Applied Example 2 – *Solution*

Puritron's monthly fixed cost is always $10,000, so it can be described by the constant function:

$$F(x) = 10{,}000$$

The variable cost can be described by the function:

$$V(x) = -0.0001x^2 + 10x$$

The total cost is the sum of the fixed cost $F$ and the variable cost $V$:

$$C(x) = V(x) + F(x)$$

$$= -0.0001x^2 + 10x + 10{,}000 \quad (0 \leq x \leq 40{,}000)$$

# Applied Example 3

Lets now consider profits

Suppose that the total revenue $R$ realized by Puritron from the sale of $x$ water filters is given by

$$R(x) = -0.0005x^2 + 20x \qquad (0 \le x \le 40{,}000)$$

Find

a. The total profit function for Puritron.

b. The total profit when Puritron produces 10,000 filters per month.

# Applied Example 3 – *Solution*

a. The total profit $P$ realized by the firm is the difference between the total revenue $R$ and the total cost $C$:

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.0005x^2 + 20x) - (-0.0001x^2 + 10x + 10{,}000) \\ &= -0.0004x^2 + 10x - 10{,}000 \end{aligned}$$

b. The total profit realized by Puritron when producing 10,000 filters per month is

$$\begin{aligned} P(x) &= -0.0004(10{,}000)^2 + 10(10{,}000) - 10{,}000 \\ &= 50{,}000 \end{aligned}$$

or \$50,000 per month.

# The Composition of Two Functions

Another way to build a function from other functions is through a process known as the composition of functions.

Consider the functions *f* and *g*:

$$f(x) = x^2 - 1 \qquad g(x) = \sqrt{x}$$

Evaluating the function *g* at the point *f*(*x*), we find that:

$$g\left(f(x)\right) = \sqrt{f(x)} = \sqrt{x^2 - 1}$$

This is an entirely new function, which we could call *h*:

$$h(x) = \sqrt{x^2 - 1}$$

# The Composition of Two Functions

Let $f$ and $g$ be functions.

Then the composition of $g$ and $f$ is the function $g \circ f$ (read "$g$ circle $f$") defined by

$$(g \circ f)(x) = g(f(x))$$

The domain of $g \circ f$ is the set of all $x$ in the domain of $f$ such that $f(x)$ lies in the domain of $g$.

# Example 4

Let $f(x) = x^2 - 1$ **and** $g(x) = \sqrt{x} + 1$.

Find:

a. The rule for the composite function $g \circ f$.

b. The rule for the composite function $f \circ g$.

Solution:

a. To find $g \circ f$, evaluate the function $g$ at $f(x)$:

$$(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} + 1 = \sqrt{x^2 - 1} + 1$$

b. To find $f \circ g$, evaluate the function $f$ at $g(x)$:

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 - 1 = (\sqrt{x} + 1)^2 - 1$$
$$= x + 2\sqrt{x} + 1 - 1 = x + 2\sqrt{x}$$

# Applied Example 5

An environmental impact study conducted for the city of Oxnard indicates that, under existing environmental protection laws, the level of carbon monoxide (CO) present in the air due to pollution from automobile exhaust will be $0.01x^{2/3}$ parts per million when the number of motor vehicles is $x$ thousand.

A separate study conducted by a state government agency estimates that $t$ years from now the number of motor vehicles in Oxnard will be $0.2t^2 + 4t + 64$ thousand.

Find:

a. An expression for the concentration of CO in the air due to automobile exhaust $t$ years from now.

b. The level of concentration 5 years from now.

# Applied Example 5(a) – *Solution*

The level of CO is described by the function

$$g(x) = 0.01x^{2/3}$$

where $x$ is the number (in thousands) of motor vehicles.

In turn, the number (in thousands) of motor vehicles is described by the function

$$f(x) = 0.2t^2 + 4t + 64$$

where $t$ is the number of years from now.

Therefore, the concentration of CO due to automobile exhaust $t$ years from now is given by

$$(g \circ f)(t) = g(f(t)) = 0.01(0.2t^2 + 4t + 64)^{2/3}$$

# Applied Example 5(b) – *Solution* cont'd

The level of CO five years from now is:

$$(g \circ f)(5) = g(f(5))$$

$$= 0.01[0.2(5)^2 + 4(5) + 64]^{2/3}$$

$$= (0.01)89^{2/3}$$

$$\approx 0.20$$

or approximately 0.20 parts per million.