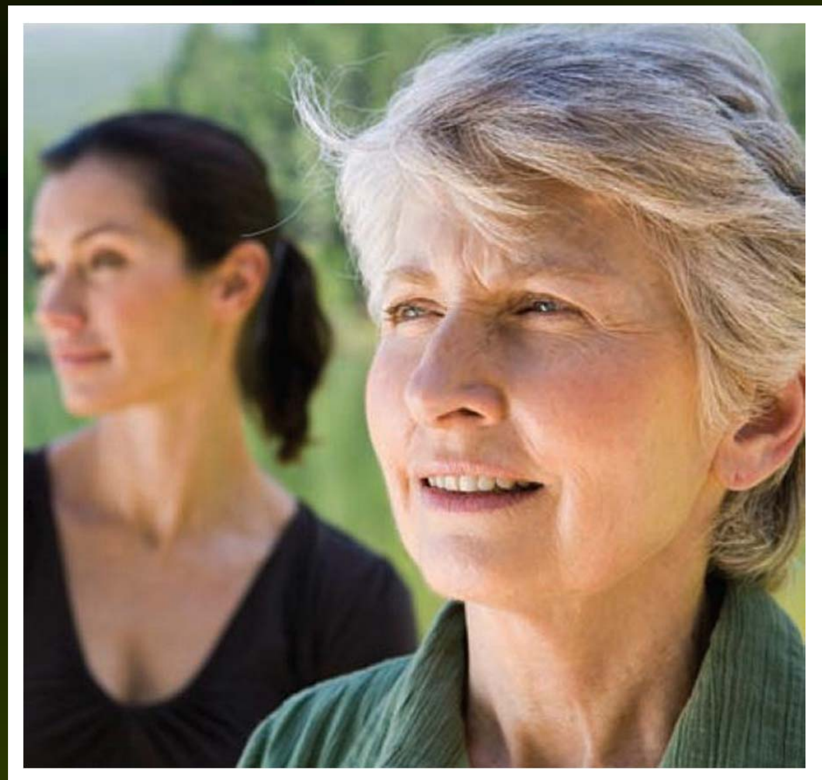
2 FUNCTIONS, LIMITS, AND THE DERIVATIVE



2.3 Functions and Mathematical Models

Mathematical Models

As we have seen, mathematics can be used to solve real-world problems.

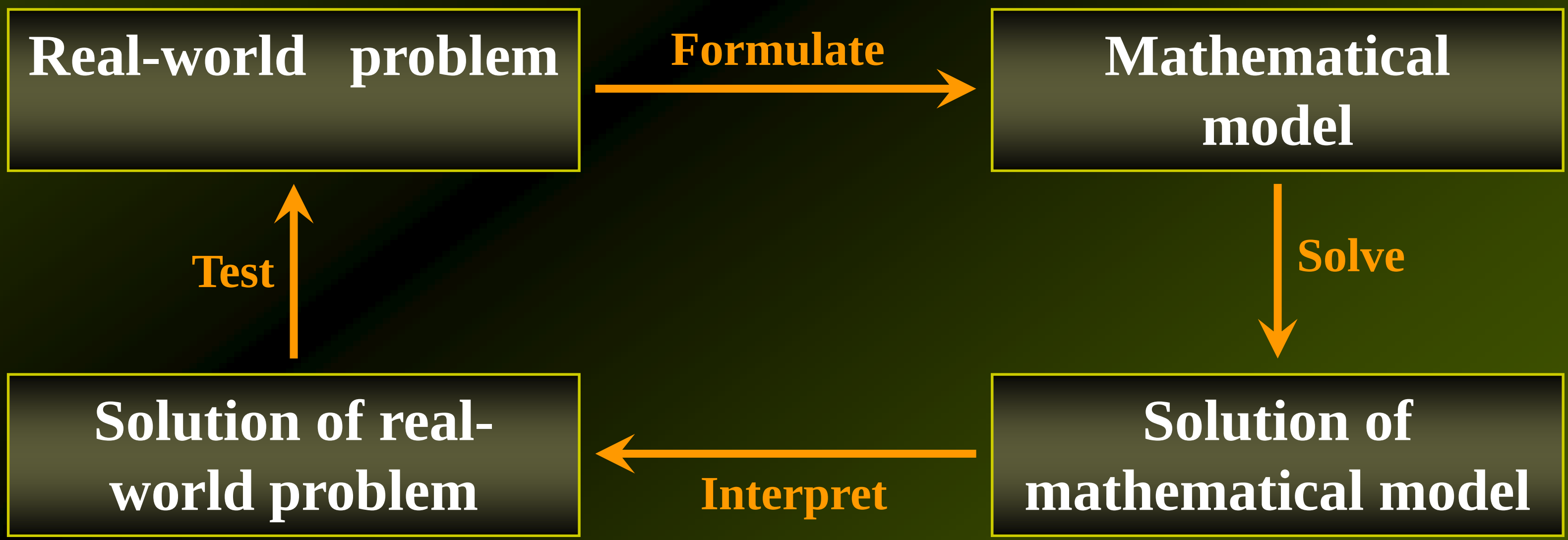
We will now discuss a few more examples of real-world phenomena, such as:

- The solvency of the U.S. Social Security trust fund (p.79)
- Global warming (p. 78)

Mathematical Modeling

Regardless of the field from which the real-world problem is drawn, the problem is analyzed using a process called mathematical modeling.

The four steps in this process are:



Modeling With Polynomial Functions

A polynomial function of degree n is a function of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \qquad (a_n \neq 0)$$

where n is a nonnegative integer and the numbers a_0, a_1, …. a_n are constants called the coefficients of the polynomial function.

Modeling With Polynomial Functions

Examples:

1. The function below is polynomial function of degree 5:

$$f(x) = 2x^5 - 3x^4 + \tfrac{1}{2}x^3 + \sqrt{2}x^2 - 6$$

2. The function below is polynomial function of degree 3:

$$g(x) = 0.001x^3 - 0.2x^2 + 10x + 200$$

Applied Example 1 – *Market for Cholesterol-Reducing Drugs*

In a study conducted in early 2000, experts projected a rise in the market for cholesterol-reducing drugs.

The U.S. market (in billions of dollars) for such drugs from 1999 through 2004 was

Year	1999	2000	2001	2002	2003	2004
Market	12.07	14.07	16.21	18.28	20.00	21.72

A mathematical model giving the approximate U.S. market over the period in question is given by

$$M(t) = 1.95t + 12.19$$

where t is measured in years, with $t = 0$ for 1999.

Applied Example 1 – *Market for Cholesterol-Reducing Drugs*

cont'd

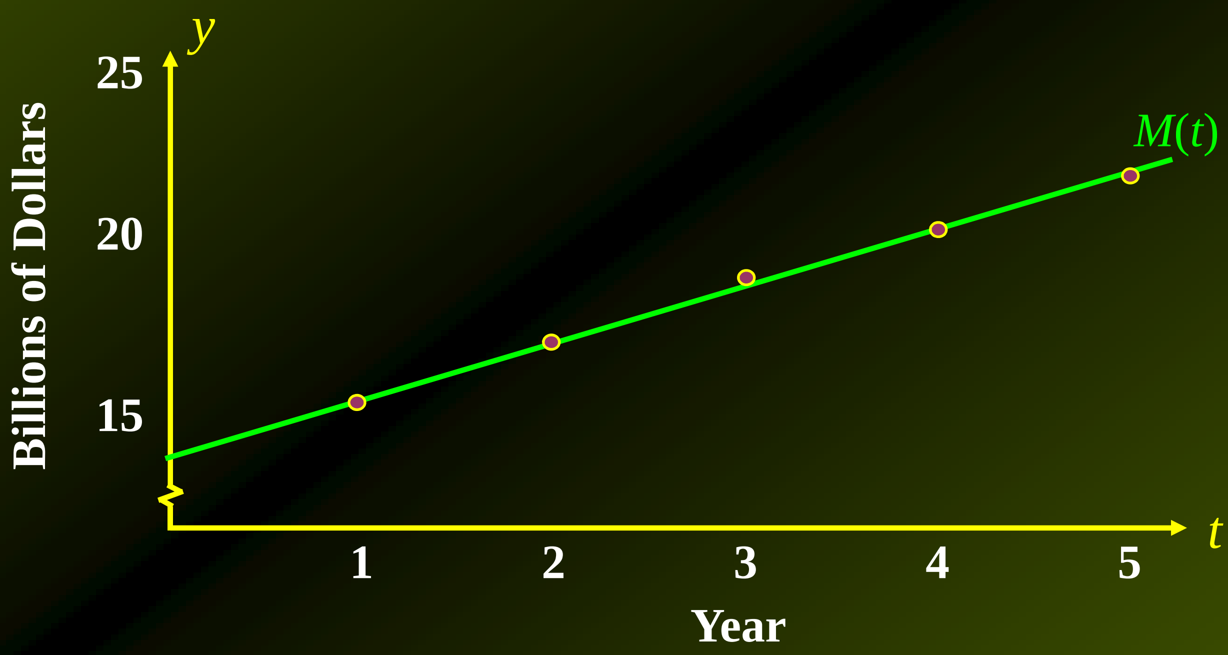
a. Sketch the graph of the function M and the given data on the same set of axes.

b. Assuming that the projection held and the trend continued, what was the market for cholesterol-reducing drugs in 2005 ($t = 6$)?

c. What was the rate of increase of the market for cholesterol-reducing drugs over the period in question?

Applied Example 1 – *Solution*

a. Graph:



Applied Example 1 – *Solution*

cont'd

b. The projected market in 2005 for cholesterol-reducing drugs was

$$M(6) = 1.95(6) + 12.19 = 23.89$$

or $23.89 billion.

c. The function M is linear, and so we see that the rate of increase of the market for cholesterol-reducing drugs is given by the slope of the straight line represented by M, which is approximately $1.95 billion per year.

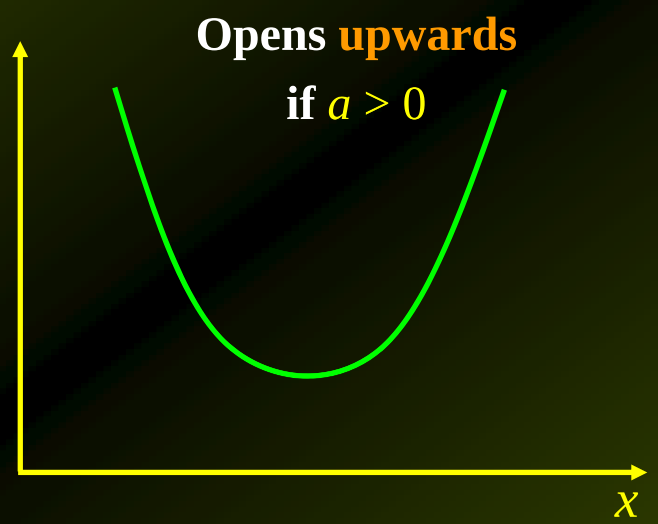
Modeling a Polynomial Function of Degree 2

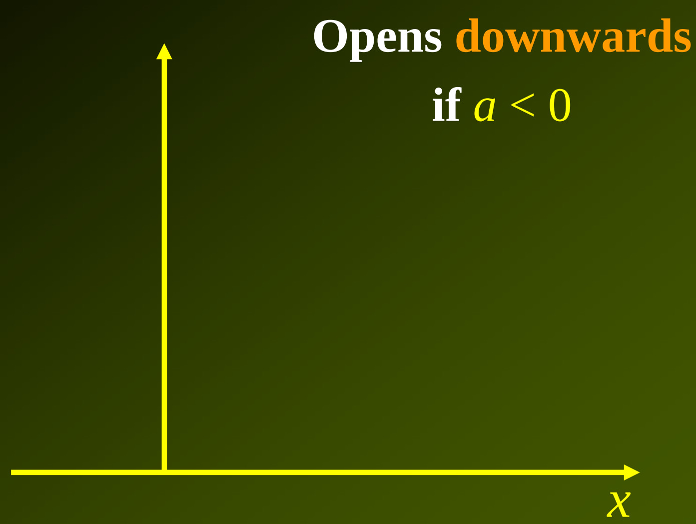
A polynomial function of degree 2 has the form

$$f(x) = a_2 x^2 + a_1 x + a_0 \qquad (a_2 \neq 0)$$

Or more simply, $y = ax^2 + bx + c$, and is called a quadratic function.

The graph of a quadratic function is a parabola:





Applied Example 2 – *Global Warming*

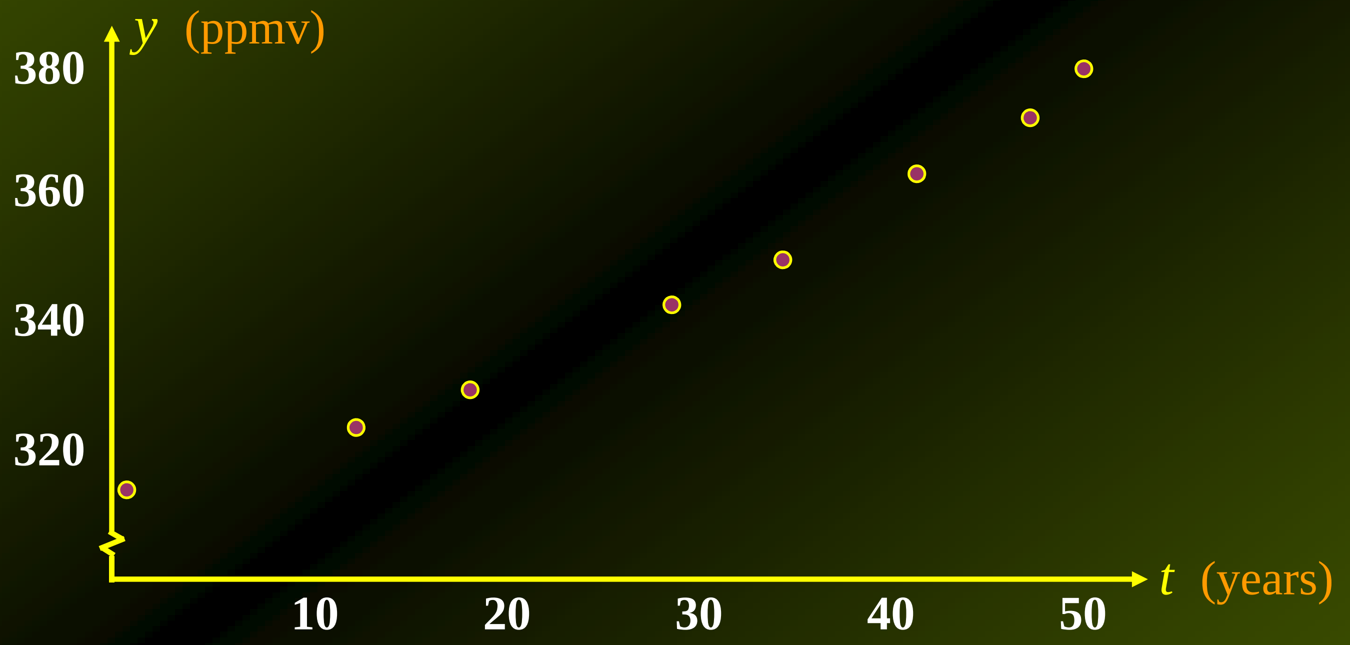
The increase in carbon dioxide (CO_2) in the atmosphere is a major cause of global warming.

Below is a table showing the average amount of CO_2, measured in parts per million volume (ppmv) for various years from **1958** through **2007**:

Year	1958	1970	1974	1978	1985	1991	1998	2003	2007
Amount	315	325	330	335	345	355	365	375	380

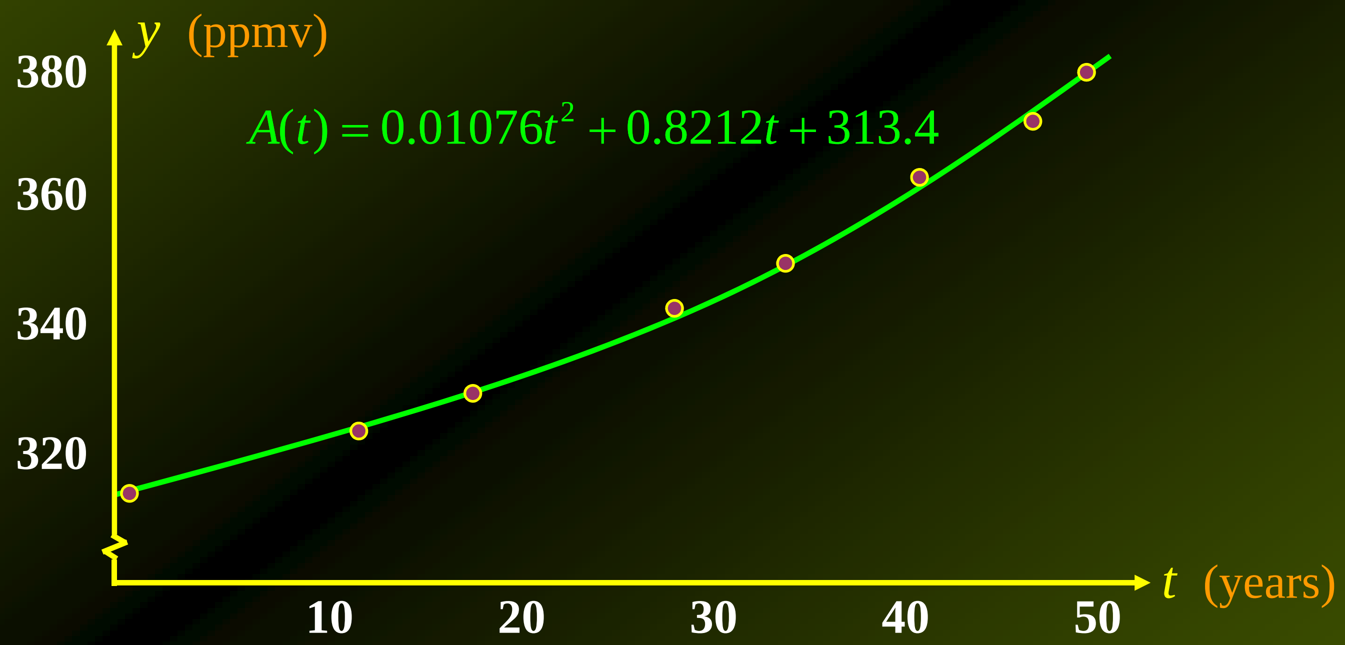
Applied Example 2 – *Global Warming* cont'd

Below is a scatter plot associated with these data:



Applied Example 2 – *Global Warming* cont'd

A mathematical model giving the approximate amount of CO_2 is given by:



Applied Example 2 – *Global Warming* cont'd

a. Use the model to estimate the average amount of atmospheric CO_2 in 1980 ($t = 23$).

b. Assume that the trend continued and use the model to predict the average amount of atmospheric CO_2 in 2010.

Applied Example 2 – *Solution*

a. The average amount of atmospheric CO_2 in 1980 is given by

$$A(23) = 0.010716(23)^2 + 0.8212(23) + 313.4 \approx 337.96$$

or approximately 338 ppmv.

b. Assuming that the trend will continue, the average amount of atmospheric CO_2 in 2010 will be

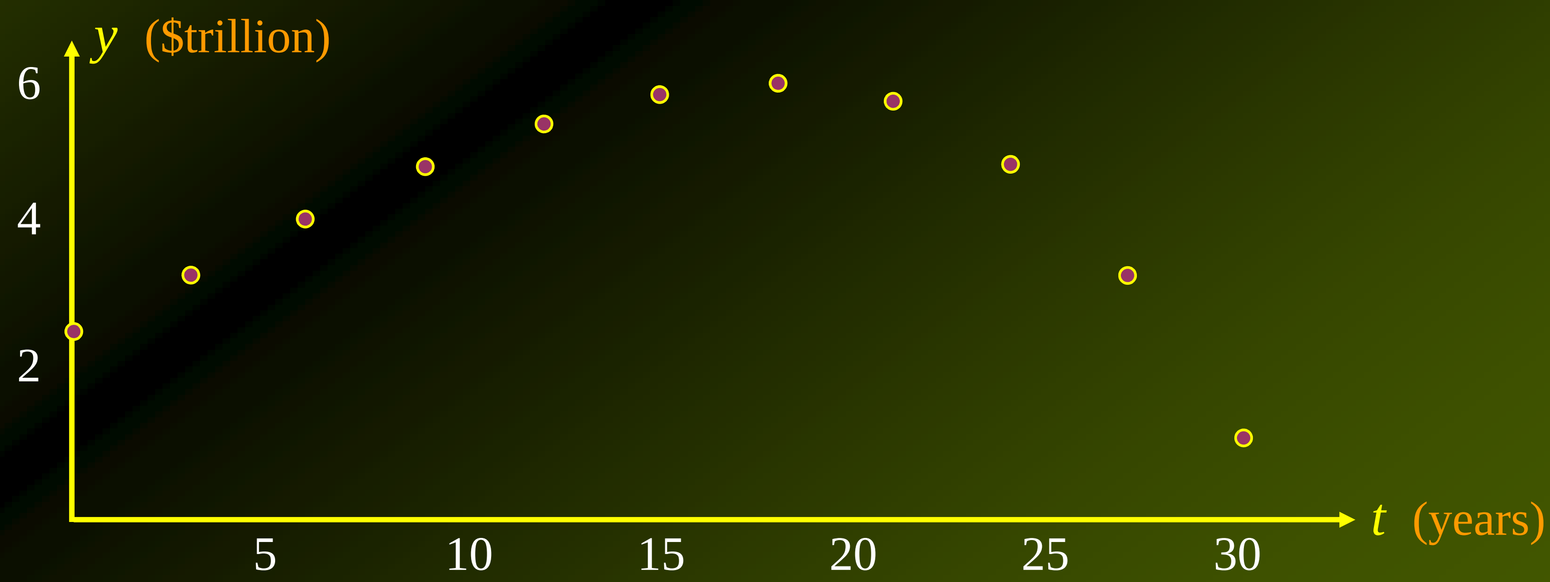
$$A(53) = 0.010716(53)^2 + 0.8212(53) + 313.4 \approx 387.03$$

Applied Example 3 – *Social Security Trust Fund Assets*

The projected assets of the Social Security trust fund (in trillions of dollars) from 2008 through 2040 are given by:

Year	2008	2011	2014	2017	2020	2023	2026	2029	2032	2035	2038	2040
Assets	2.4	3.2	4.0	4.7	5.3	5.7	5.9	5.6	4.9	3.6	1.7	0

The scatter plot associated with these data is:

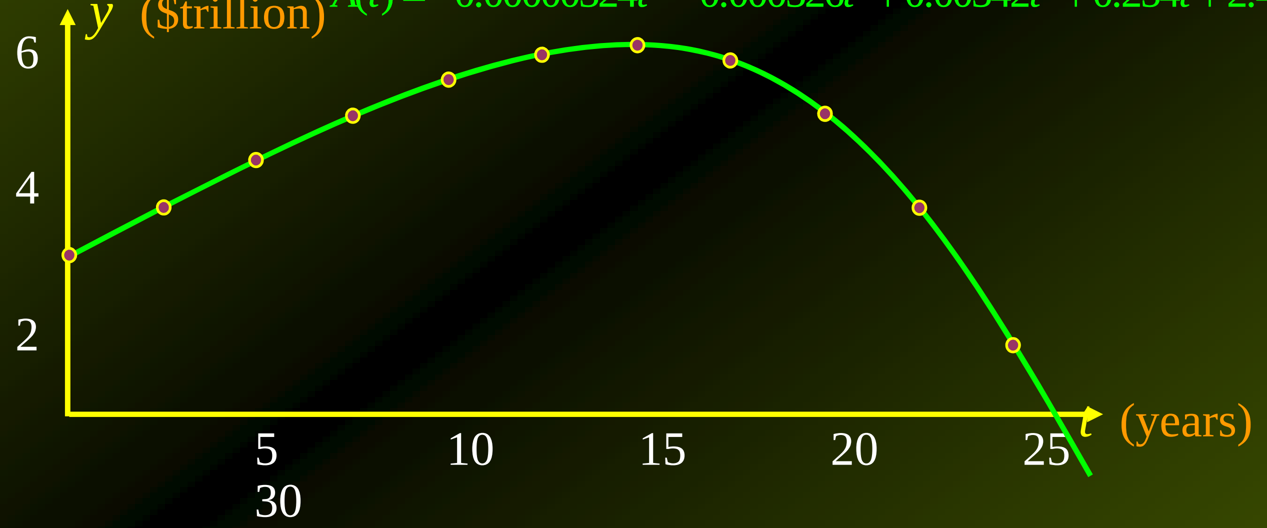


Applied Example 3 – *Social Security Trust Fund Assets*

cont'd

A mathematical model giving the approximate value of assets in the trust fund (in trillions of dollars) is:

$$A(t) = -0.00000324t^4 - 0.000326t^3 + 0.00342t^2 + 0.254t + 2.4$$



Applied Example 3 – *Social Security Trust Fund Assets*

cont'd

a. The first baby boomers will turn 65 in 2011. What will be the assets of the Social Security trust fund at that time?

b. The last of the baby boomers will turn 65 in 2029. What will the assets of the trust fund be at the time?

c. Use the graph of function $A(t)$ to estimate the year in which the current Social Security system will go broke.

Applied Example 3 – *Solution*

a. The assets of the Social Security fund in 2011 ($t = 3$) will be:

$$A(3) = -0.00000324(3)^4 - 0.000326(3)^3 + 0.00342(3)^2 + 0.254(3) + 2.4 \approx 3.18$$

or approximately $3.18 trillion.

b. The assets of the Social Security fund in 2029 ($t = 21$) will be:

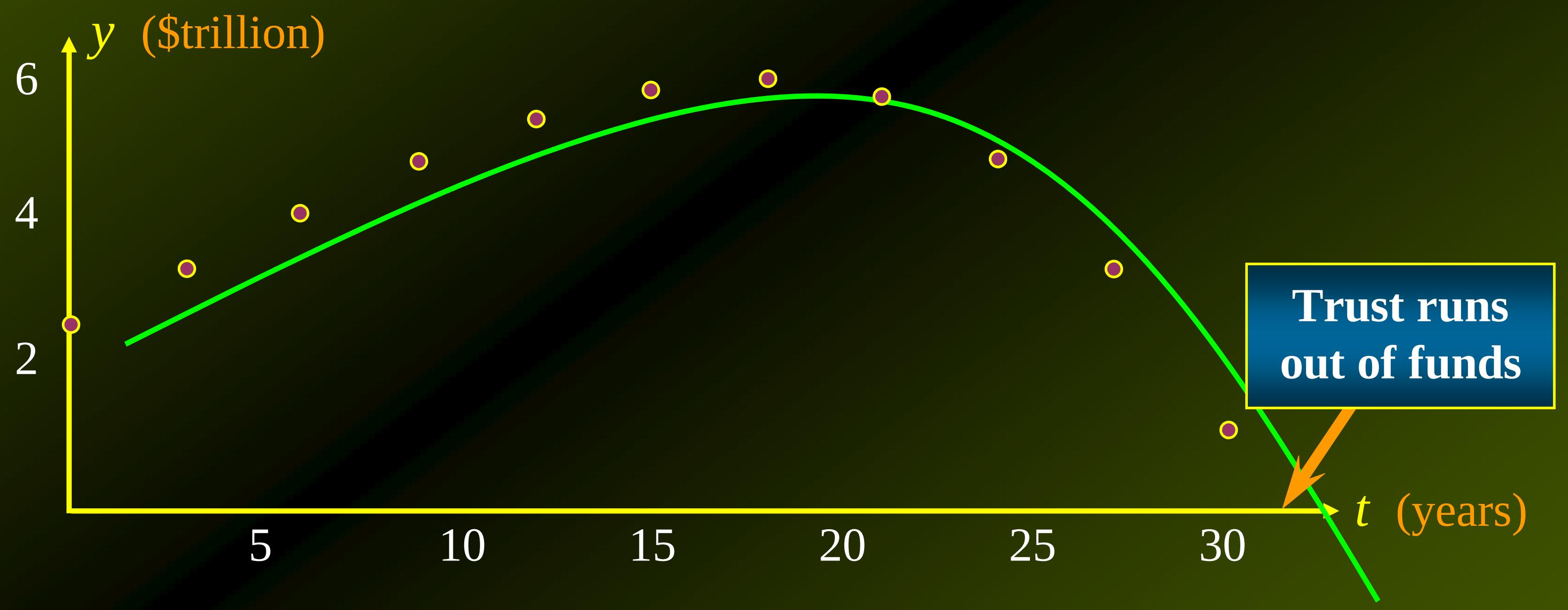
$$A(21) = -0.00000324(21)^4 - 0.000326(21)^3 + 0.00342(21)^2 + 0.254(21) + 2.4 \approx 5.59$$

or approximately $5.59 trillion.

Applied Example 3 – *Solution*

cont'd

c. The graph shows that function *A* crosses the *t*-axis at about $t = 32$, suggesting the system will go broke by 2040:



Rational and Power Functions

A *rational function* is simply the quotient of two polynomials.

In general, a rational function has the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x)$ and $g(x)$ are polynomial functions.

Since the division by zero is not allowed, we conclude that the domain of a rational function is the set of all real numbers except the zeros of g (the roots of the equation $g(x) = 0$)

Rational and Power Functions

Examples of rational functions:

$$F(x) = \frac{3x^3 + x^2 - x + 1}{x - 2}$$

$$G(x) = \frac{x^2 + 1}{x^2 - 1}$$

Rational and Power Functions

Functions of the form

$$f(x) = x^r$$

where *r* is any real number, are called *power functions*.

We encountered examples of power functions earlier in our work.

Examples of power functions:

$$f(x) = \sqrt{x} = x^{1/2} \qquad \textbf{and} \qquad g(x) = \frac{1}{x^2} = x^{-2}$$

Rational and Power Functions

Many functions involve combinations of rational and power functions.

Examples:

$$f(x) = \sqrt{\frac{1-x^2}{1+x^2}}$$

$$g(x) = \sqrt{x^2 - 3x + 4}$$

$$h(x) = (1+2x)^{1/2} + \frac{1}{(x^2+2)^{3/2}}$$

Applied Example 4 – *Driving Costs*

A study of driving costs based on a 2007 medium-sized sedan found the following average costs (car payments, gas, insurance, upkeep, and depreciation), measured in cents per mile:

Miles/year, x	5000	10,000	15,000	20,000
Cost/mile, y (¢)	83.8	62.9	52.2	47.1

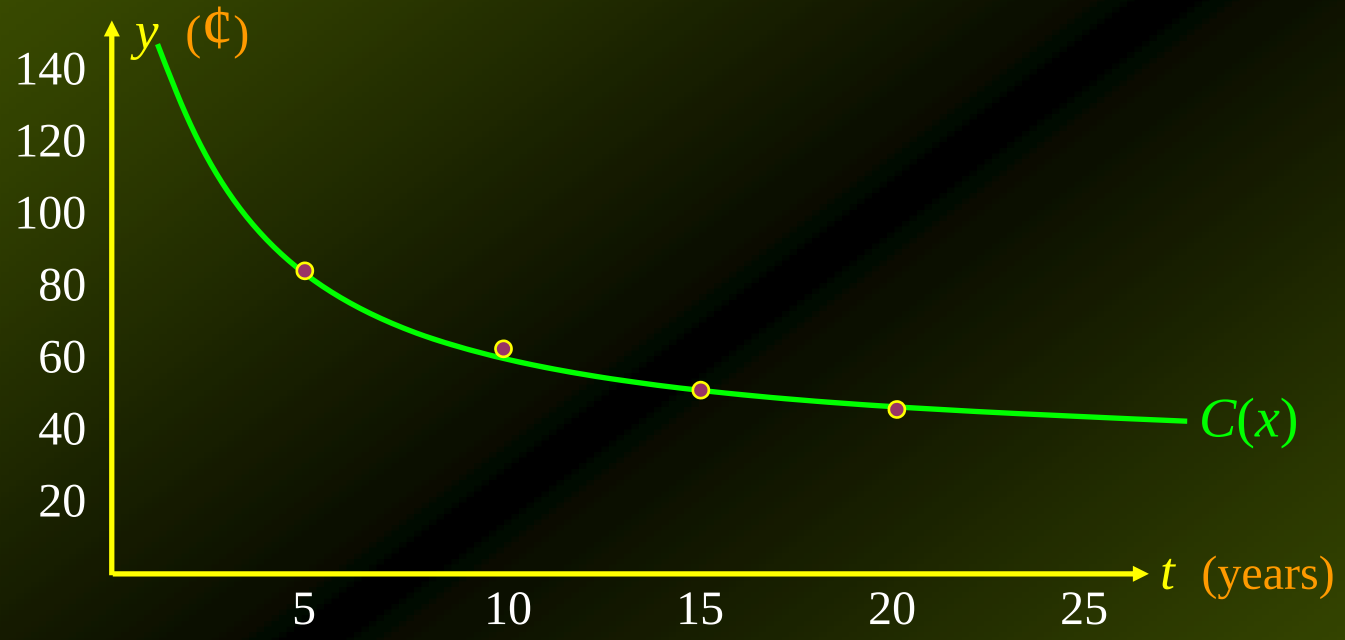
A mathematical model giving the average cost in cents per mile is:

$$C(x) = \frac{164.8}{x^{0.42}}$$

where x (in thousands) denotes the number of miles the car is driven in 1 year.

Applied Example 4 – *Driving Costs* cont'd

Below is the scatter plot associated with this data:



Using this model, estimate the average cost of driving a 2007 medium-sized sedan 8,000 miles per year and 18,000 miles per year.

Applied Example 4 – *Solution*

The average cost for driving a car 8,000 miles per year is

$$C(8) = \frac{164.8}{(8)^{0.42}} \approx 68.81$$

or approximately 68.8¢/mile.

The average cost for driving a car 18,000 miles per year is

$$C(18) = \frac{164.8}{(18)^{0.42}} \approx 48.95$$

or approximately 48.95¢/mile.

Some Economic Models

People's decision on how much to *demand* or purchase of a given product depends on the price of the product:

- The higher the price, the less they want to buy of it.
- A demand function $p = d(x)$ can be used to describe this.

Some Economic Models

Similarly, firms' decision on how much to *supply* or produce of a product depends on the price of the product:

- The higher the price, the more they want to produce of it.
- A supply function $p = s(x)$ can be used to describe this.

Some Economic Models

The interaction between demand and supply will ensure the market settles to a *market equilibrium*:

- This is the situation at which quantity demanded equals quantity supplied.

- Graphically, this situation occurs when the demand curve and the supply curve intersect: where $d(x) = s(x)$.

Applied Example 5 – *Supply and Demand*

The demand function for a certain brand of bluetooth wireless headset is given by

$$p = d(x) = -0.025x^2 - 0.5x + 60$$

The corresponding supply function is given by

$$p = s(x) = 0.02x^2 + 0.6x + 20$$

where *p* is the expressed in dollars and *x* is measured in units of a thousand. Find the equilibrium quantity and price.

Applied Example 5 – *Solution*

We solve the following system of equations:

$$p = -0.025x^2 - 0.5x + 60$$
$$p = 0.02x^2 + 0.6x + 20$$

Substituting the second equation into the first yields:

$$0.02x^2 + 0.6x + 20 = -0.025x^2 - 0.5x + 60$$

$$0.045x^2 + 1.1x - 40 = 0$$

$$45x^2 + 1100x - 40,000 = 0$$

$$9x^2 + 220x - 8,000 = 0$$

$$(9x + 400)(x - 20) = 0$$

Applied Example 5 – *Solution*

cont'd

Thus, either $x = –400/9$ (but this is not possible), or $x = 20$.

So, the equilibrium quantity must be 20,000 headsets.

The equilibrium price is given by:

$$p = 0.02(20)^2 + 0.6(20) + 20 = 40$$

or $40 per headset.

Constructing Mathematical Models

Some mathematical models can be constructed using elementary geometric and algebraic arguments.

Guidelines for constructing mathematical models:

1. Assign a letter to each variable mentioned in the problem. If appropriate, draw and label a figure.
2. Find an expression for the quantity sought.
3. Use the conditions given in the problem to write the quantity sought as a function *f* of one variable.

   Note any restrictions to be placed on the domain of *f* by the nature of the problem.

Applied Example 6 – *Enclosing an Area*

The owner of the Rancho Los Feliz has 3000 yards of fencing with which to enclose a rectangular piece of grazing land along the straight portion of a river. Fencing is not required along the river.

Letting x denote the width of the rectangle, find a function f in the variable x giving the area of the grazing land if she uses all of the fencing.

Applied Example 6 – *Solution*

This information was given:

1. The area of the rectangular grazing land is $A = xy$.

2. The amount of fencing is $2x + y$ which must equal 3000 (to use all the fencing), so:

$$2x + y = 3000$$

Solving for y we get:

$$y = 3000 - 2x$$

Substituting this value of y into the expression for A gives:

$$A = x(3000 - 2x) = 3000x - 2x^2$$

Applied Example 6 – *Solution*

cont'd

Finally, x and y represent distances, so they must be nonnegative, so $x \geq 0$ and $y = 3000 - 2x \geq 0$ (or $x \leq 1500$).

Thus, the required function is:

$$f(x) = 3000x - 2x^2 \qquad (0 \leq x \leq 1500)$$

Applied Example 7 – *Charter-Flight Revenue*

If exactly 200 people sign up for a charter flight, Leasure World Travel Agency charges $300 per person. However, if more than 200 people sign up for the flight (assume this is the case), then each fare is reduced by $1 for each additional person.

Letting x denote the number of passengers above 200, find a function giving the revenue realized by the company.

Applied Example 7 – *Solution*

This information was given.

1. If there are x passengers above 200, then the number of passengers signing up for the flight is $200 + x$.

2. The fare will be $(300 - x)$ dollars per passenger.

The revenue will be

$$R = (200 + x)(300 - x)$$

$$= -x^2 + 100x + 60{,}000$$

Applied Example 7 – *Solution*

cont'd

The quantities must be positive, so $x \geq 0$ and $300 - x \geq 0$ (or $x \leq 300$).

So the required function is:

$$f(x) = -x^2 + 100x + 60{,}000 \qquad (0 \leq x \leq 300)$$