2

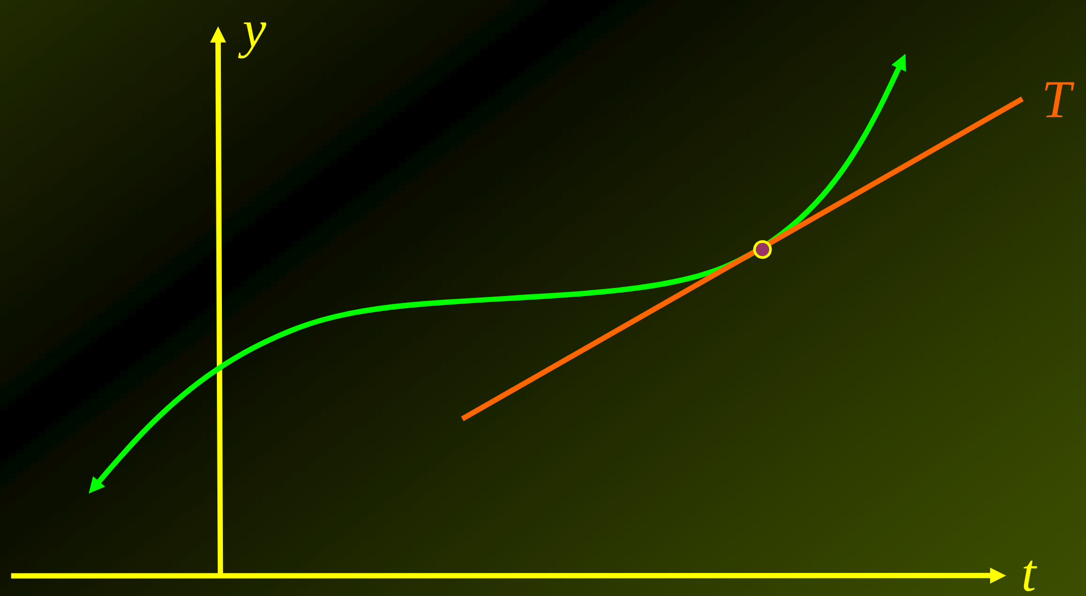
FUNCTIONS, LIMITS, AND THE DERIVATIVE

2.4 Limits

Introduction to Calculus

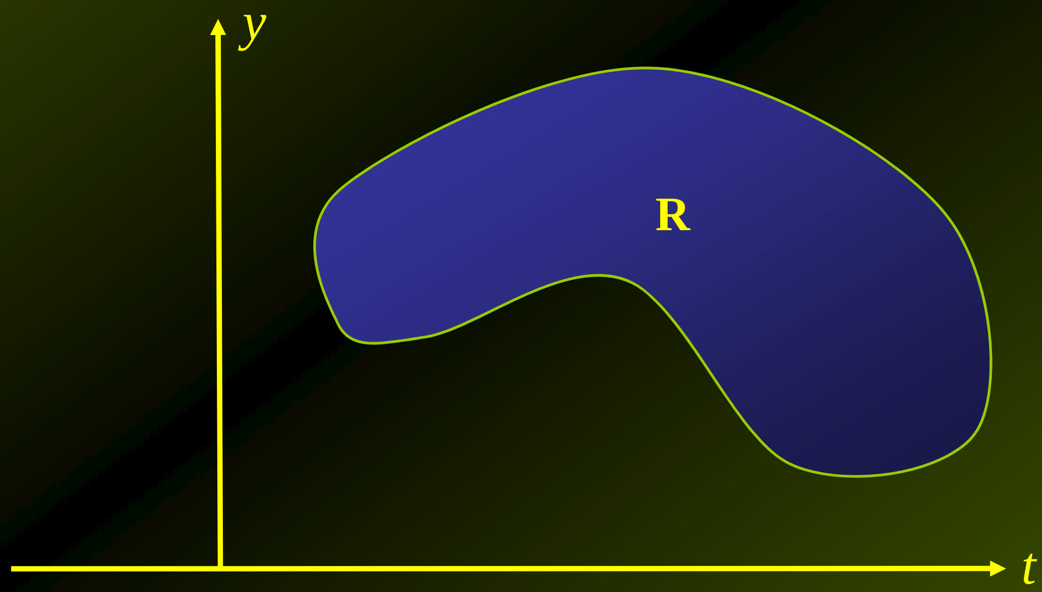
Historically, the development of calculus by Isaac Newton and Gottfried W. Leibniz resulted from the investigation of the following problems:

1. Finding the tangent line to a curve at a given point on the curve:



Introduction to Calculus

2. Finding the area of planar region bounded by an arbitrary curve.



Introduction to Calculus

The study of the tangent-line problem led to the creation of *differential calculus*, which relies on the concept of the derivative of a function.

The study of the area problem led to the creation of *integral calculus*, which relies on the concept of the anti-derivative, or integral, of a function.

Example – *A Speeding Maglev*

From data obtained in a test run conducted on a prototype of maglev, which moves along a straight monorail track, engineers have determined that the position of the maglev (in feet) from the origin at time t is given by

$$s = f(t) = 4t^2 \qquad (0 \le t \le 30)$$

Where f is called the position function of the maglev.

The position of the maglev at time $t = 0, 1, 2, 3, \ldots, 10$ is

$f(0) = 0$ $\quad f(1) = 4$ $\quad f(2) = 16$ $\quad f(3) = 36 \ldots$ $\quad f(10) = 400$

But what if we want to find the *velocity* of the maglev at any given point in time?

Example – *A Speeding Maglev*

Say we want to find the *velocity* of the maglev at $t = 2$.

We may compute the average velocity of the maglev over an interval of time, such as [2, 4] as follows:

$$\frac{\text{Distance covered}}{\text{Time elapsed}} = \frac{f(4) - f(2)}{4 - 2}$$

$$= \frac{4(4^2) - 4(2^2)}{2}$$

$$= \frac{64 - 16}{2}$$

$$= 24$$

or 24 feet/second.

Example – *A Speeding Maglev*

This is not the velocity of the maglev at exactly $t = 2$, but it is a useful *approximation*.

We can find a better approximation by choosing a *smaller interval* to compute the speed, say [2, 3].

More generally, let $t > 2$. Then, the average velocity of the maglev over the time interval $[2, t]$ is given by

$$\frac{\text{Distance covered}}{\text{Time elapsed}} = \frac{f(t) - f(2)}{t - 2}$$

Example – *A Speeding Maglev*

$$= \frac{4(t^2) - 4(2^2)}{t - 2}$$

$$= \frac{4(t^2 - 4)}{t - 2}$$

$$\text{Average velocity} = \frac{4(t^2 - 4)}{t - 2}$$

By choosing the values of *t* closer and closer to 2, we obtain average velocities of the maglev over smaller and smaller time intervals.

Example – *A Speeding Maglev*

The smaller the time interval, the closer the average velocity becomes to the *instantaneous velocity* of the train at $t = 2$, as the table below demonstrates:

t	2.5	2.1	2.01	2.001	2.0001
Average Velocity	18	16.4	16.04	16.004	16.0004

The closer t gets to 2, the closer the average velocity gets to 16 feet/second.

Thus, the instantaneous velocity at $t = 2$ seems to be 16 feet/second.

Intuitive Definition of a Limit

Consider the function *g,* which gives the average velocity of the maglev:

$$g(t) = \frac{4(t^2 - 4)}{t - 2}$$

Suppose we want to find the value that $g(t)$ approaches as t approaches 2.

- We take values of t approaching 2 from the right (as we did before), and we find that $g(t)$ approaches 16:

t	2.5	2.1	2.01	2.001	2.0001
$g(t)$	18	16.4	16.04	16.004	16.0004

Intuitive Definition of a Limit

- Similarly, we take values of t approaching 2 from the left, and we find that $g(t)$ also approaches 16:

t	1.5	1.9	1.99	1.999	1.9999
$g(t)$	14	15.6	15.96	15.996	15.9996

Intuitive Definition of a Limit

We have found that as t approaches 2 from either side, $g(t)$ approaches 16.

In this situation, we say that *the limit* of $g(t)$ as t approaches 2 is 16.

This is written as

$$\lim_{t \to 2} g(t) = \lim_{t \to 2} \frac{4(t^2 - 4)}{t - 2} = 16$$

Observe that $t = 2$ is not in the domain of $g(t)$.

But this does not matter, since $t = 2$ does not play any role in computing this limit.

Limit of a Function

The function *f* has a limit *L* as *x* approaches *a*, written

$$\lim_{x \to a} f(x) = L$$

If the value of *f*(*x*) can be made as close to the number *L* as we please by taking *x* values sufficiently close to (but not equal to) *a*.

Example 1

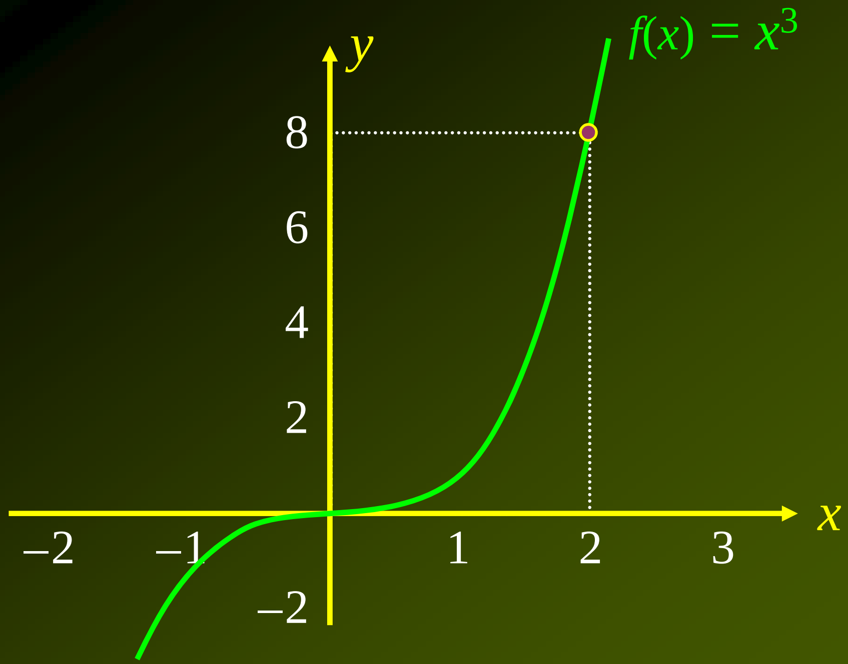
Let $f(x) = x^3$. Evaluate $\lim_{x \to 2} f(x)$.

Solution:

You can see in the graph that $f(x)$ can be as close to 8 as we please by taking x sufficiently close to 2.

Therefore,

$$\lim_{x \to 2} x^3 = 8$$



Example 2

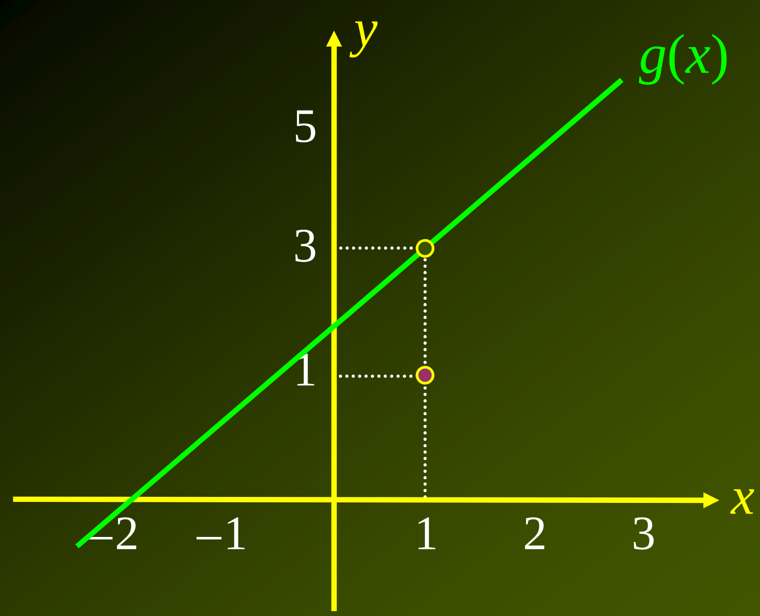
Let $g(x) = \begin{cases} x+2 & \textbf{if } x \neq 1 \\ 1 & \textbf{if } x = 1 \end{cases}$. Evaluate $\lim_{x \to 1} g(x)$.

Solution:

You can see in the graph that $g(x)$ can be as close to 3 as we please by taking x sufficiently close to 1.

Therefore,

$$\lim_{x \to 1} g(x) = 3$$

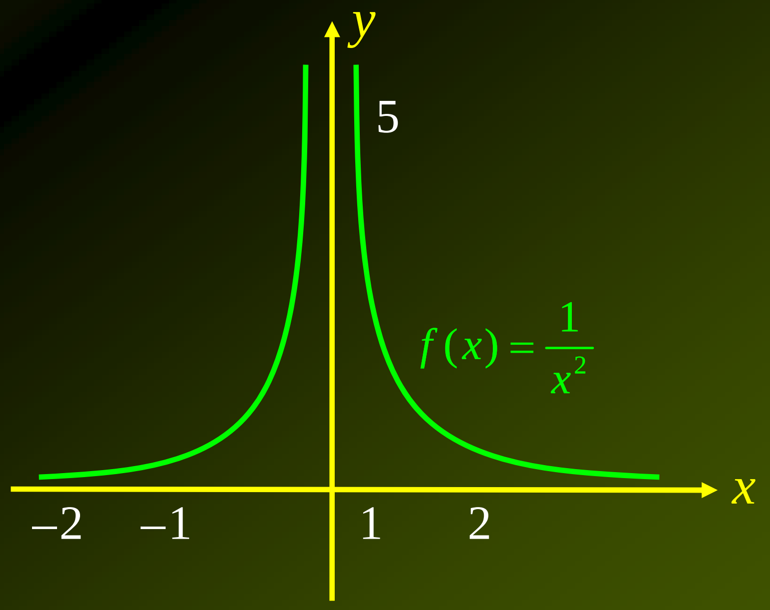


Example 3(b)

Let $f(x) = \frac{1}{x^2}$. Evaluate $\lim_{x \to 0} f(x)$.

Solution:

The graph shows us that as x approaches 0 from either side, $f(x)$ increases without bound and thus does not approach any specific real number.



Thus, the limit of $f(x)$ does not exist as x approaches 0.

Theorem 1: Properties of Limits

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$

Then,

1. $\lim_{x\to a} [f(x)]^r = \left[\lim_{x\to a} f(x)\right]^r = L^r$ *r*, a real number
2. $\lim_{x\to a} cf(x) = c\lim_{x\to a} f(x) = cL$ *c*, a real number
3. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm M$
4. $\lim_{x\to a} [f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$ Provided that $M \neq 0$

Example 4

Use theorem 1 to evaluate the following limits:

a. $\lim_{x\to 2} x^3 = \left[\lim_{x\to 2} x\right]^3 = 2^3 = 8$

b. $\lim_{x\to 4} 5x^{3/2} = 5\left[\lim_{x\to 4} x\right]^{3/2} = 5(4)^{3/2} = 40$

c. $\lim_{x\to 1} (5x^4 - 2) = 5\left[\lim_{x\to 1} x\right]^4 - \lim_{x\to 1} 2 = 5(1)^4 - 2 = 3$

Example 4

cont'd

d. $\lim_{x \to 3} 2x^3 \sqrt{x^2 + 7} = 2\left[\lim_{x \to 3} x\right]^3 \sqrt{\lim_{x \to 3} x^2 + 7} = 2(3)^3 \sqrt{(3)^2 + 7} = 216$

e. $\lim_{x \to 2} \dfrac{2x^2 + 1}{x + 1} = \dfrac{\lim_{x \to 2} (2x^2 + 1)}{\lim_{x \to 2} (x + 1)} = \dfrac{2(2)^2 + 1}{2 + 1} = \dfrac{9}{3} = 3$

Indeterminate Forms

Let's consider $\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2}$ which we evaluated earlier for the maglev example by looking at values for *x* near *x* = 2.

If we attempt to evaluate this expression by applying Property 5 of limits, we get

$$\lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2} = \frac{\lim_{x \to 2} 4(x^2 - 4)}{\lim_{x \to 2} x - 2} = \frac{0}{0}$$

In this case we say that the limit of the quotient *f*(*x*)/*g*(*x*) as *x* approaches 2 has the indeterminate form 0/0.

This expression does *not* provide us with a solution to our problem.

Strategy for Evaluating Indeterminate Forms

1. Replace the given function with an appropriate one that takes on the same values as the original function everywhere except at $x = a$.

2. Evaluate the limit of this function as x approaches a.

Example 5

Evaluate $\lim_{x \to 2} \frac{4(x^2-4)}{x-2}$

Solution:

As we've seen, here we have an indeterminate form 0/0.

We can rewrite

$$\frac{4(x^2-4)}{x-2} = \frac{4(x-2)(x+2)}{x-2} = 4(x+2) \qquad x \neq 2$$

Thus, we can say that

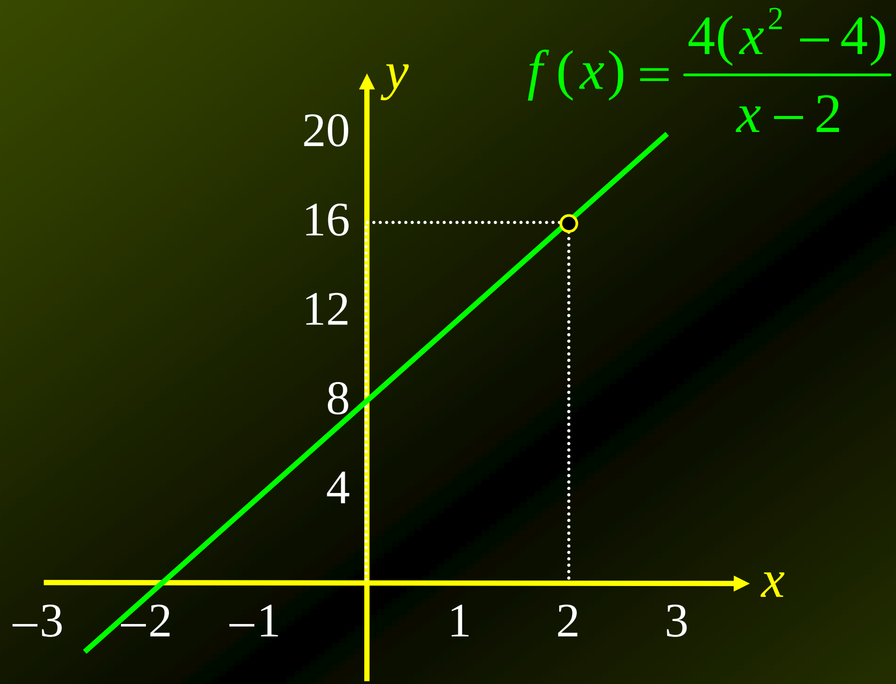
$$\lim_{x \to 2} \frac{4(x^2-4)}{x-2} = \lim_{x \to 2} 4(x+2) = 16$$

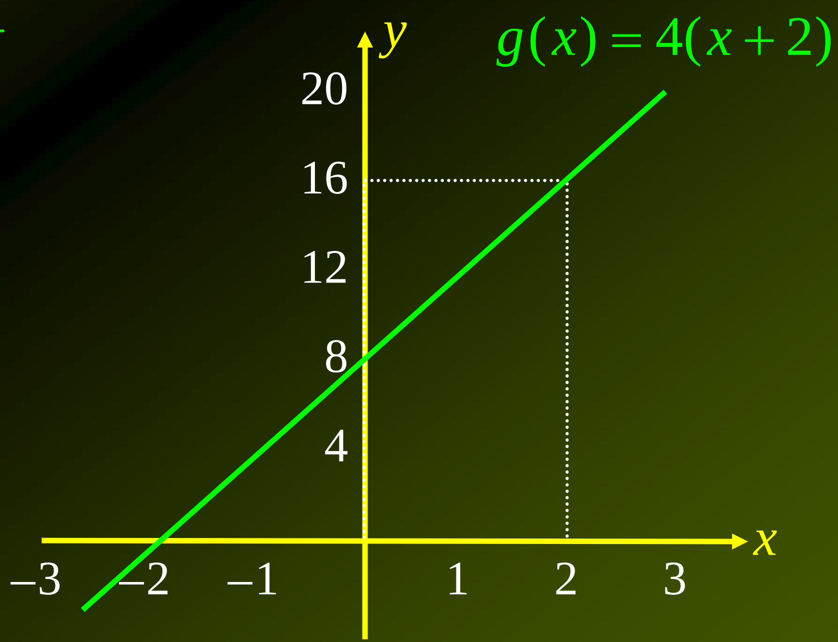
Note that 16 is the same value we obtained for the maglev example through approximation.

Example 5 – *Solution*

cont'd

Notice in the graphs below that the two functions yield the same graphs, except for the value $x = 2$:





Example 6

Evaluate $\lim_{h\to 0}\dfrac{\sqrt{1+h}-1}{h}$

Solution:

As we've seen, here we have an indeterminate form 0/0.

We can rewrite (with the constraint that $h \neq 0$):

$$\frac{\sqrt{1+h}-1}{h}=\frac{\sqrt{1+h}-1}{h}\cdot\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}=\frac{h}{h(\sqrt{1+h}+1)}=\frac{1}{\sqrt{1+h}+1}$$

Thus, we can say that

$$\lim_{h\to 0}\frac{\sqrt{1+h}-1}{h}=\lim_{h\to 0}\frac{1}{\sqrt{1+h}+1}=\frac{1}{\sqrt{1}+1}=\frac{1}{2}$$

Limits at Infinity

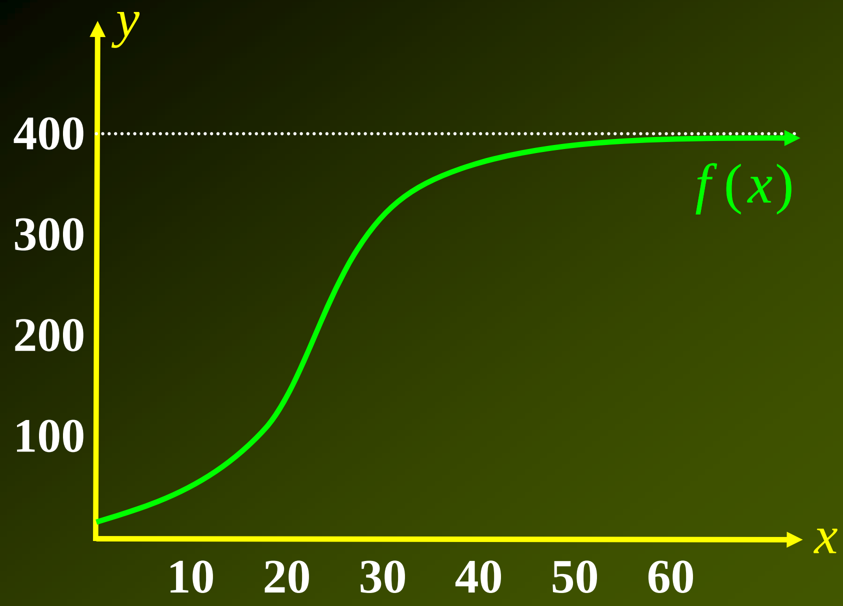
There are occasions when we want to know whether $f(x)$ approaches a unique number as x increases without bound.

In the graph below, as x increases without bound, $f(x)$ approaches the number 400. We call the line $y = 400$ a horizontal asymptote.

In this case, we can say that

$$\lim_{x \to \infty} f(x) = 400$$

and we call this a limit of a function at infinity.



Example

Consider the function $f(x) = \frac{2x^2}{1+x^2}$

Determine what happens to $f(x)$ as x gets larger and larger.

Solution:

We can pick a sequence of values of x and substitute them in the function to obtain the following values:

x	1	2	5	10	100	1000
$f(x)$	1	1.6	1.92	1.98	1.9998	1.999998

As x gets larger and larger, $f(x)$ gets closer and closer to 2.

Thus, we can say that

$$\lim_{x \to \infty} \frac{2x^2}{1+x^2} = 2$$

Limit of a Function at Infinity

- The function f has the limit L as x increases without bound (as x approaches infinity), written

$$\lim_{x \to \infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough.

- Similarly, the function f has the limit M as x decreases without bound (as x approaches negative infinity), written

$$\lim_{x \to -\infty} f(x) = M$$

if $f(x)$ can be made arbitrarily close to M by taking x large enough in absolute value.

Example 7(a)

Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

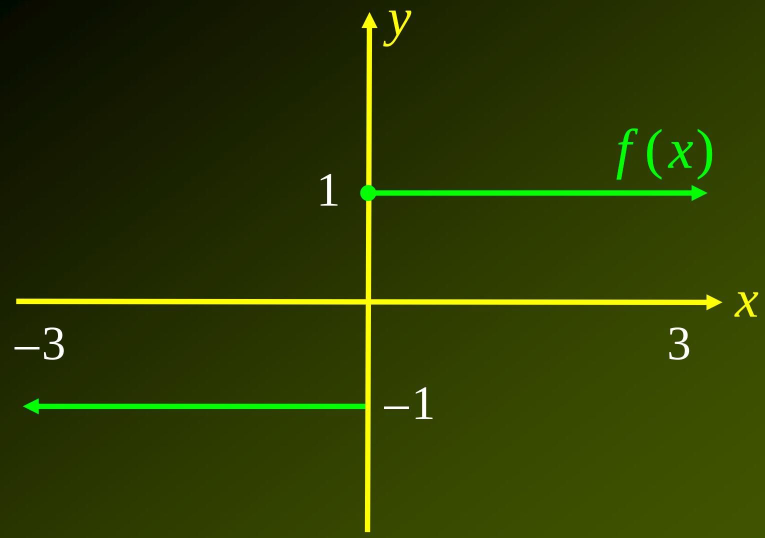
Evaluate $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$

Solution:

Graphing *f*(*x*) reveals that

$$\lim_{x \to \infty} f(x) = 1$$

$$\lim_{x \to -\infty} f(x) = -1$$



Example 7(b)

Let $g(x) = \dfrac{1}{x^2}$

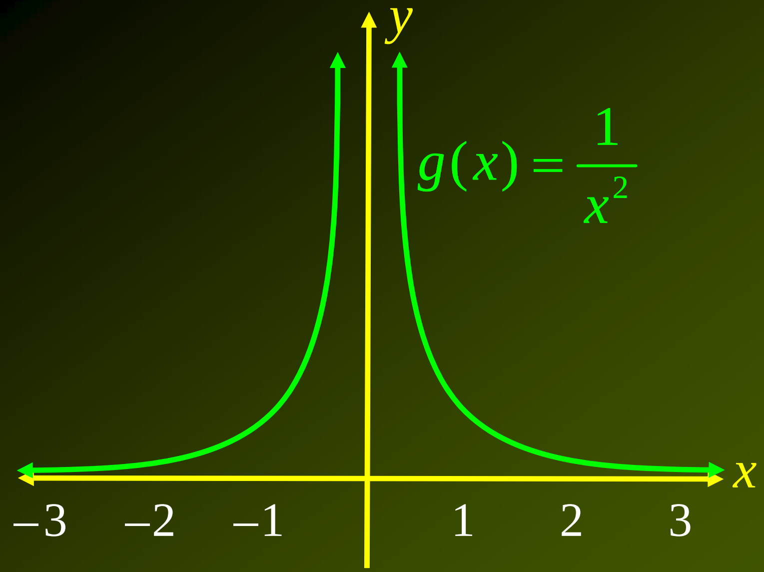
Evaluate $\lim_{x \to \infty} g(x)$ and $\lim_{x \to -\infty} g(x)$

Solution:

Graphing *g*(*x*) reveals that

$$\lim_{x \to \infty} g(x) = 0$$

$$\lim_{x \to -\infty} g(x) = 0$$



Theorem 2: Properties of Limits

All properties of limits listed in Theorem 1 are valid when *a* is replaced by ∞ or $-\infty$.

In addition, we have the following properties for limits to infinity:

For all $n > 0$, $\lim_{x \to \infty} \frac{1}{x^n} = 0$ and $\lim_{x \to -\infty} \frac{1}{x^n} = 0$

provided that $\frac{1}{x^n}$ is defined.

Example 8

Evaluate $\lim_{x\to\infty} \dfrac{x^2 - x + 3}{2x^3 + 1}$

Solution:

The limits of both the numerator and denominator do not exist as *x* approaches infinity, so property 5 is *not* applicable.

We can find the solution instead by dividing numerator and denominator by x^3:

$$\lim_{x\to\infty} \frac{(x^2 - x + 3)/x^3}{(2x^3 + 1)/x^3} = \lim_{x\to\infty} \frac{\dfrac{1}{x} - \dfrac{1}{x^2} + \dfrac{3}{x^3}}{2 + \dfrac{1}{x^3}} = \frac{0 - 0 + 0}{2 + 0} = \frac{0}{2} = 0$$

Example 9

Evaluate $\lim_{x\to\infty} \frac{3x^2+8x-4}{2x^2+4x-5}$

Solution:

Again, we see that property 5 does *not* apply.

So we divide numerator and denominator by x^2:

$$\lim_{x\to\infty} \frac{(3x^2+8x-4)/x^2}{(2x^2+4x-5)/x^2} = \lim_{x\to\infty} \frac{3+\frac{8}{x}-\frac{4}{x^2}}{2+\frac{4}{x}-\frac{5}{x^2}} = \frac{3+0-0}{2+0-0} = \frac{3}{2}$$

Example 10

Evaluate $\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4}$

Solution:

Again, we see that property 5 does *not* apply.

But dividing numerator and denominator by x^2 does *not* help in this case:

$$\lim_{x\to\infty} \frac{(2x^3 - 3x^2 + 1)/x^2}{(x^2 + 2x + 4)/x^2} = \lim_{x\to\infty} \frac{2x - 3 + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{4}{x^2}}$$

In other words, the limit does not exist. We indicate this by writing

$$\lim_{x\to\infty} \frac{2x^3 - 3x^2 + 1}{x^2 + 2x + 4} = \infty$$