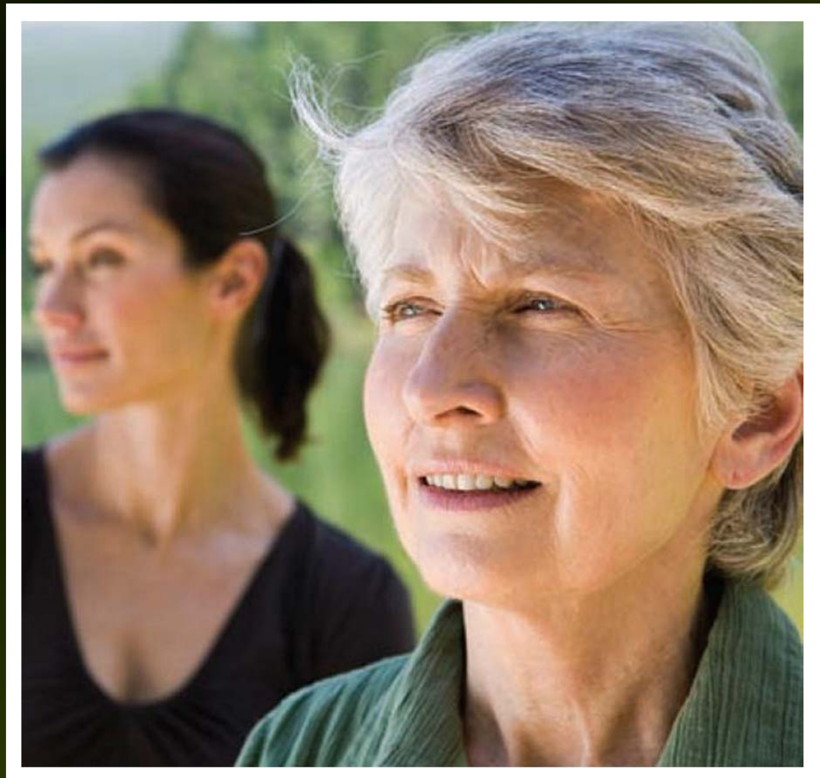
2 FUNCTIONS, LIMITS, AND THE DERIVATIVE



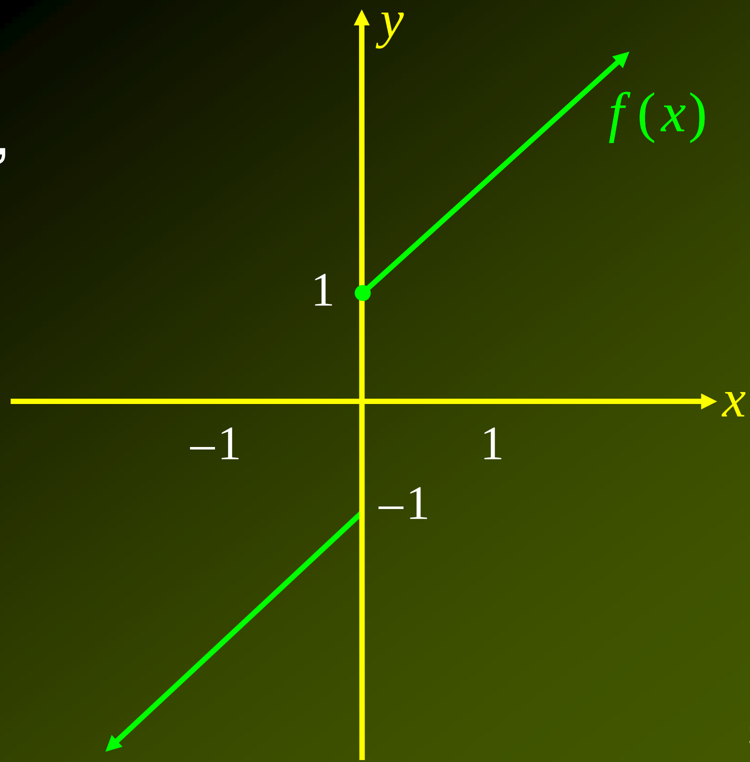
2.5 One-Sided Limits and Continuity

One-Sided Limits

Consider the function

$$f(x) = \begin{cases} x - 1 & \text{if } x < 0 \\ x + 1 & \text{if } x \geq 0 \end{cases}$$

Its graph shows that *f* does *not* have a limit as *x* approaches zero, because approaching from each side results in different values.

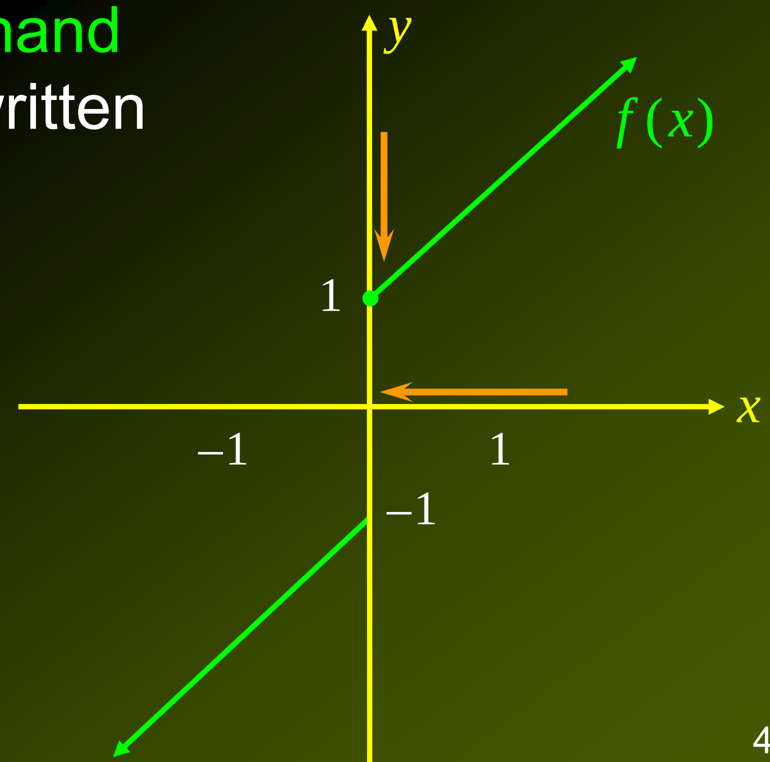


One-Sided Limits

If we restrict x to be *greater* than zero (to the right of zero), we see that $f(x)$ approaches 1 as close to as we please as x approaches 0.

In this case we say that the right-hand limit of f as x approaches 0 is 1, written

$$\lim_{x \to 0^+} f(x) = 1$$

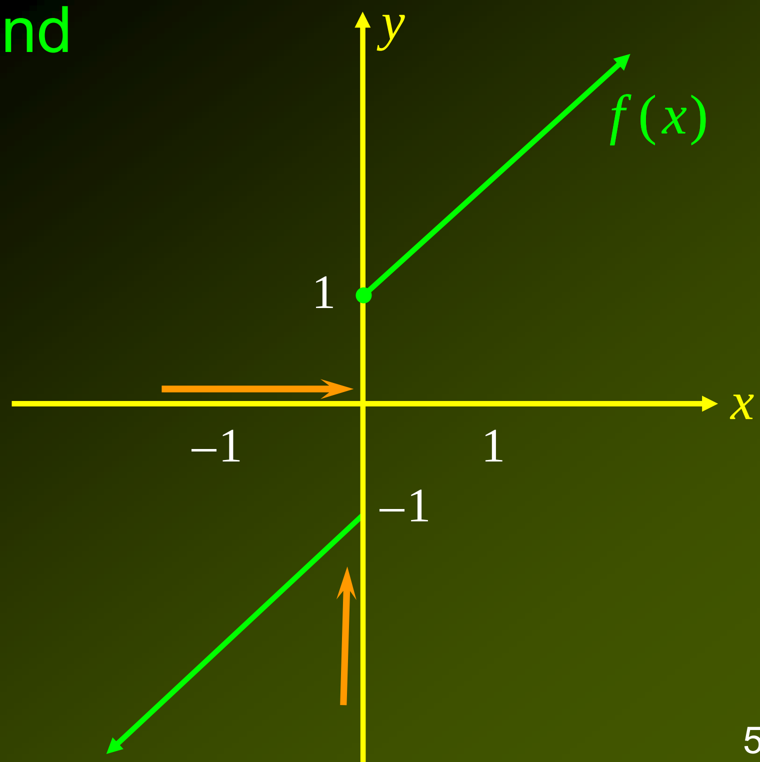


One-Sided Limits

Similarly, if we restrict *x* to be *less than zero* (to the left of zero), we see that *f*(*x*) approaches –1 as close to as we please as *x* approaches 0.

In this case we say that the left-hand limit of *f* as *x* approaches 0 is – 1, written

$$\lim_{x \to 0^-} f(x) = -1$$



One-Sided Limits

- The function *f* has the right-hand limit *L* as *x* approaches from the right, written

$$\lim_{x \to a^+} f(x) = L$$

If the values of *f*(*x*) can be made as close to *L* as we please by taking *x* sufficiently close to (but not equal to) *a* and to the right of *a*.

- Similarly, the function *f* has the left-hand limit *L* as *x* approaches from the left, written

$$\lim_{x \to a^-} f(x) = L$$

If the values of *f*(*x*) can be made as close to *L* as we please by taking *x* sufficiently close to (but not equal to) *a* and to the left of *a*.

Theorem 3: Properties of Limits

The connection between one-side limits and the two-sided limit defined earlier is given by the following theorem.

Let f be a function that is defined for all values of x close to $x = a$ with the possible exception of a itself. Then

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Example 1(a)

Show that $\lim_{x\to 0} f(x)$ exists by studying the one-sided limits of *f* as *x* approaches 0:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x > 0 \\ -x & \text{if } x \le 0 \end{cases}$$

Solution:

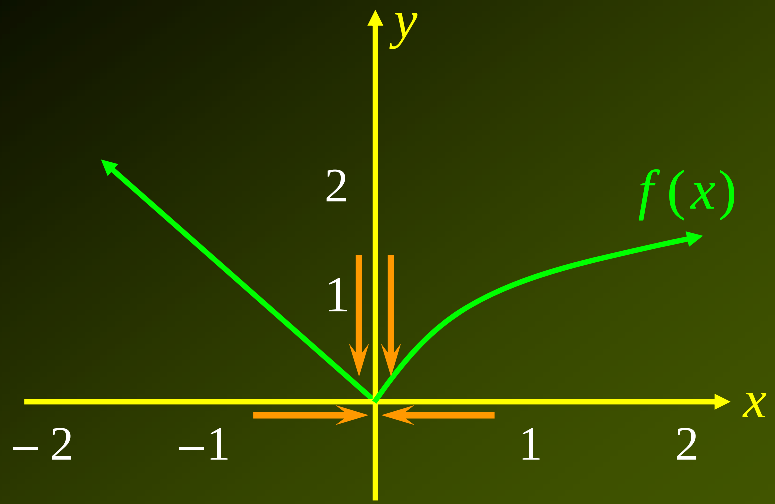
For $x > 0$, we find

$$\lim_{x\to 0^+} f(x) = 0$$

And for $x \le 0$, we find

$$\lim_{x\to 0^-} f(x) = 0$$

Thus, $\lim_{x\to 0} f(x) = 0$



Example 1(b)

Show that $\lim_{x \to 0} g(x)$ does not exist.

$$g(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Solution:

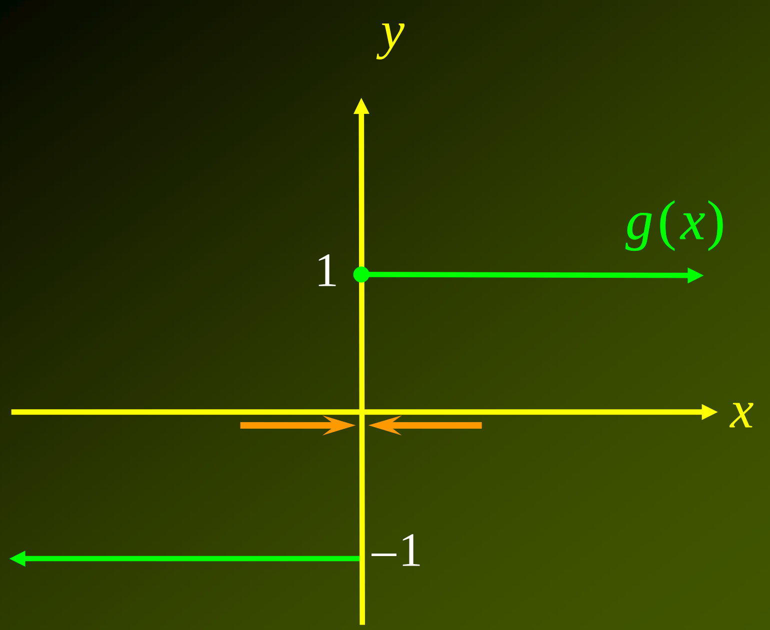
For $x < 0$, we find

$$\lim_{x \to 0^-} g(x) = -1$$

And for $x \geq 0$, we find

$$\lim_{x \to 0+} g(x) = 1$$

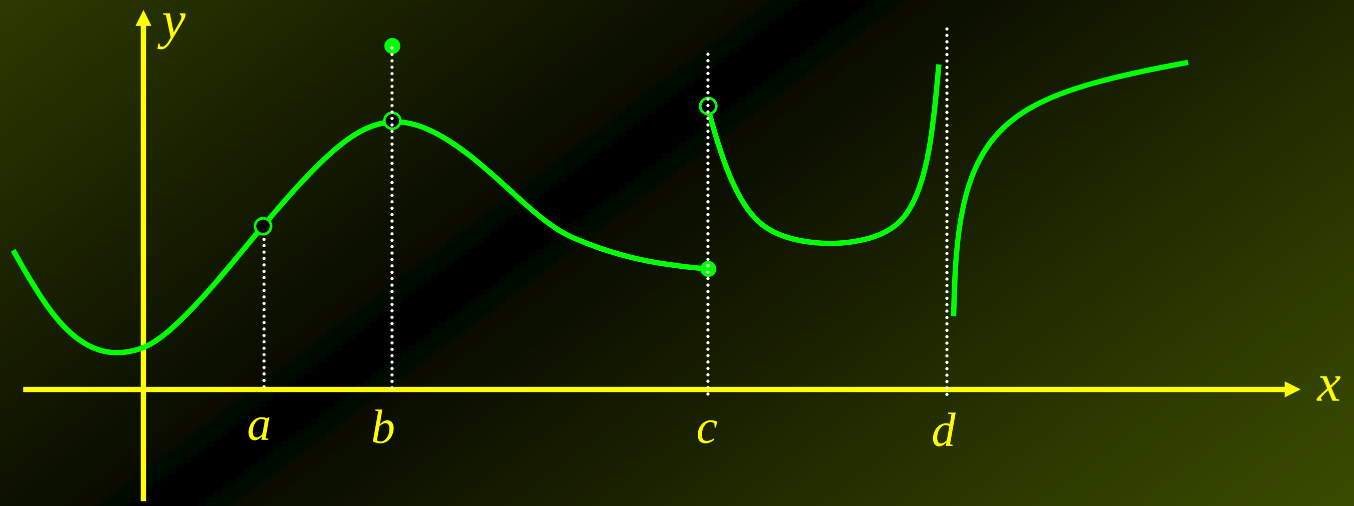
Thus, $\lim_{x \to 0} g(x)$ does not exist.



Continuous Functions

Loosely speaking, a function is continuous at a given point if its graph at that point has no holes, gaps, jumps, or breaks.

Consider, for example, the graph of *f*



This function is discontinuous at the following points:

1. At $x = a$, f is not defined ($x = a$ is not in the domain of f).

Continuous Functions

2. At $x = b$, $f(b)$ is not equal to the limit of $f(x)$ as x approaches b.

3. At $x = c$, the function does not have a limit, since the left-hand and right-hand limits are not equal.

4. At $x = d$, the limit of the function does not exist, resulting in a break in the graph.

Continuity of a Function at a Number

- A function f is continuous at a number $x = a$ if the following conditions are satisfied:
  1. $f(a)$ is defined.
  2. $\lim_{x \to a} f(x)$ **exists.**
  3. $\lim_{x \to a} f(x) = f(a)$
- If f is not continuous at $x = a$, then f is said to be discontinuous at $x = a$.
- Also, f is continuous on an interval if f is continuous at every number in the interval.

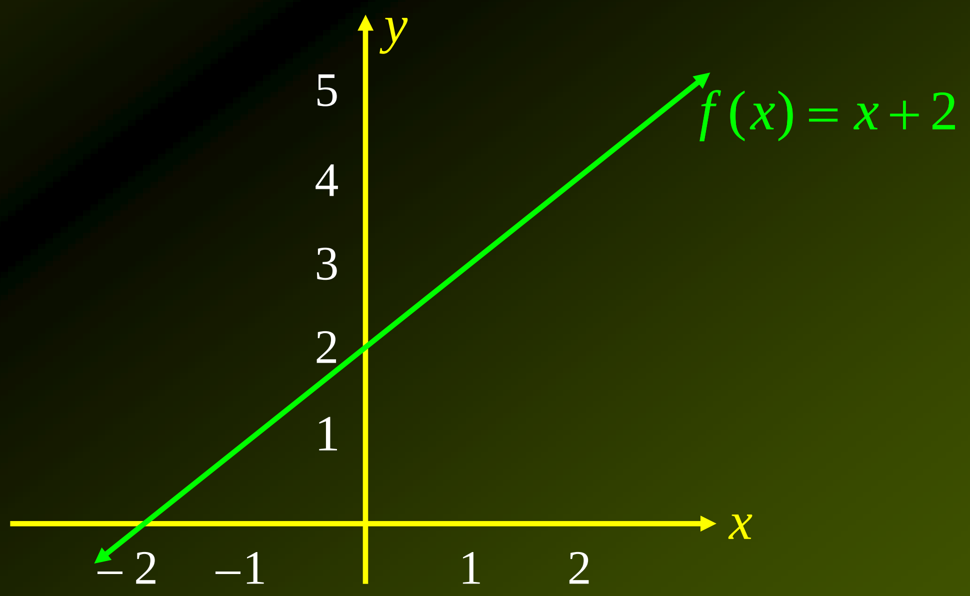
Example 2(a)

Find the values of *x* for which the function is continuous:

$$f(x) = x + 2$$

Solution:

The function *f* is continuous everywhere because the three conditions for continuity are satisfied for all values of *x*.



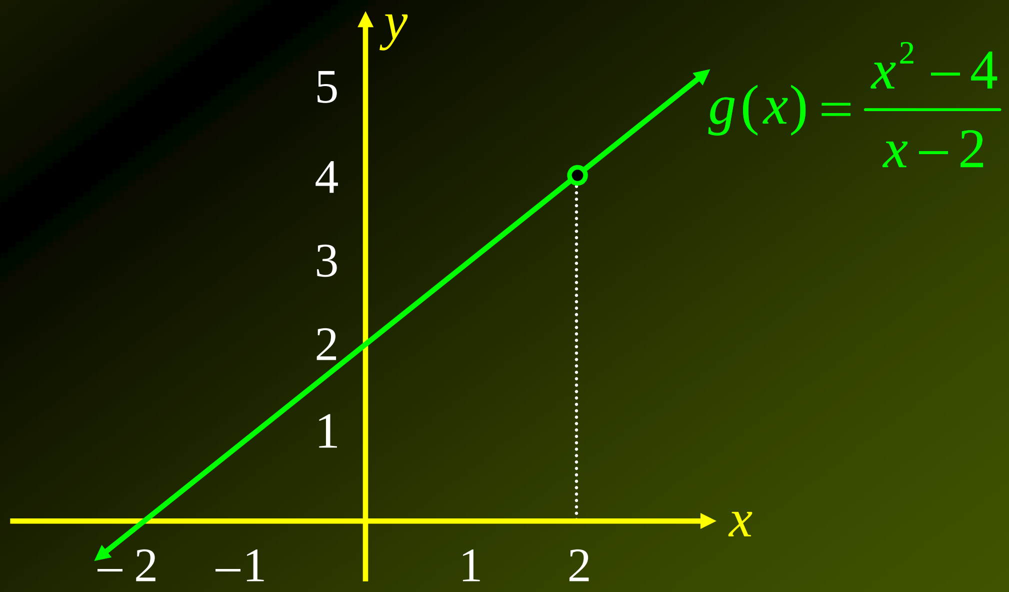
Example 2(b)

Find the values of *x* for which the function is continuous:

$$g(x) = \frac{x^2 - 4}{x - 2}$$

Solution:

The function *g* is discontinuous at *x* = 2 because *g* is not defined at that number. It is continuous everywhere else.



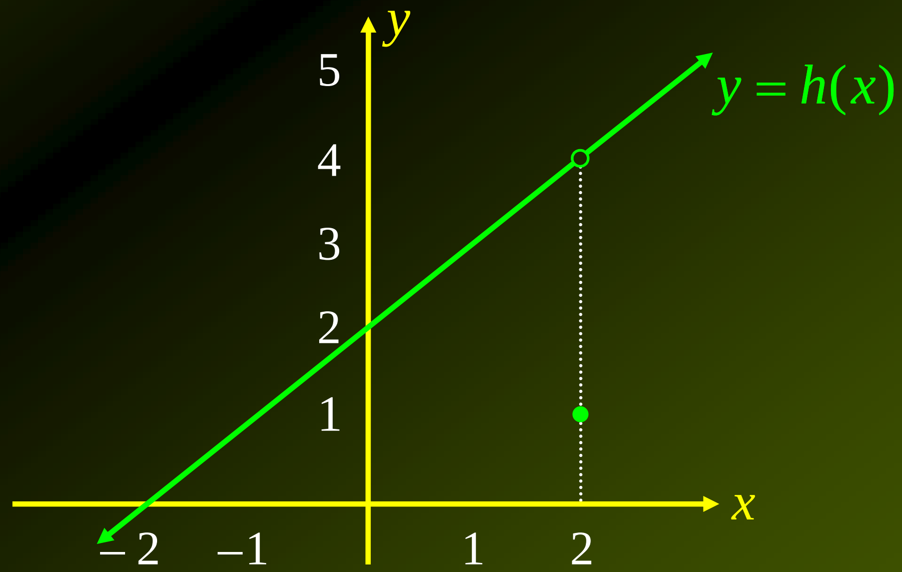
Example 2(c)

Find the values of x for which the function is continuous:

$$h(x) = \begin{cases} x+2 & \textbf{if } x \neq 2 \\ 1 & \textbf{if } x = 2 \end{cases}$$

Solution:

The function h is continuous everywhere except at $x = 2$ where it is discontinuous because $h(2) = 1 \neq \lim_{x \to 2} h(x) = 4$



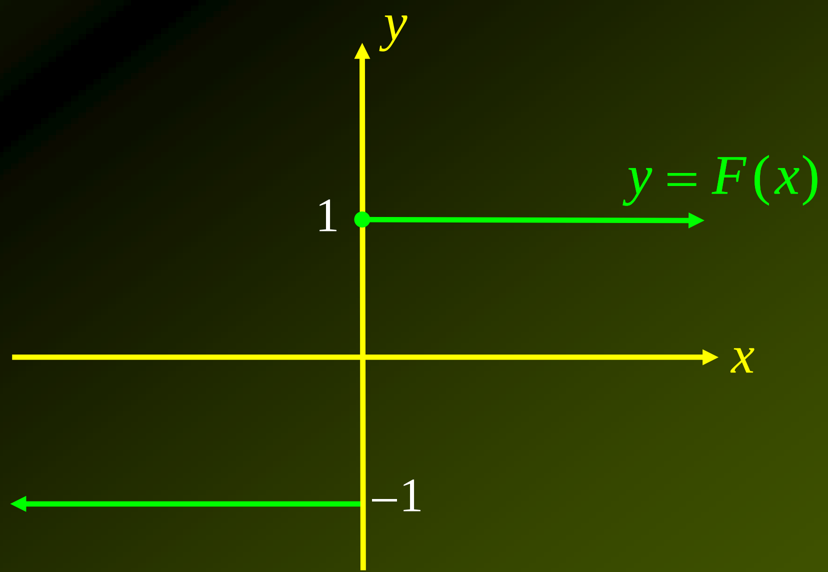
Example 2(d)

Find the values of *x* for which the function is continuous:

$$F(x) = \begin{cases} -1 & \textbf{if } x < 0 \\ 1 & \textbf{if } x \geq 0 \end{cases}$$

Solution:

The function *F* is discontinuous at *x* = 0 because the limit of *F* fails to exist as *x* approaches 0. It is continuous everywhere else.



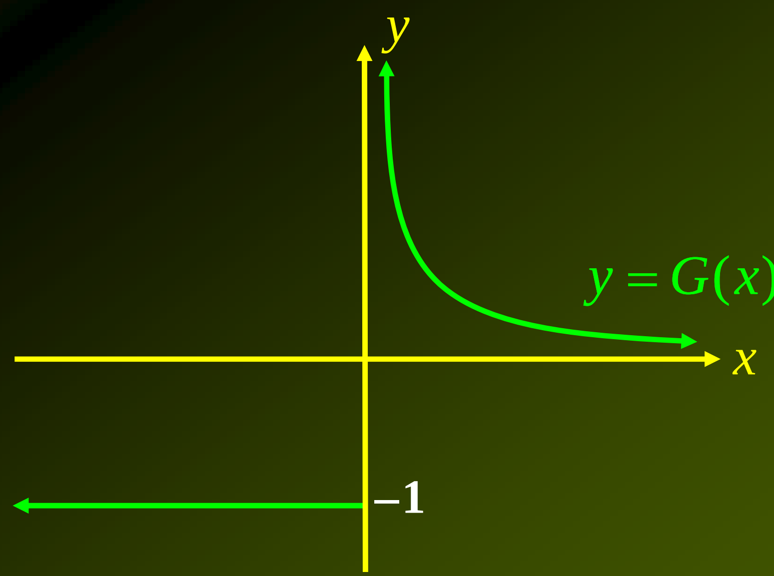
Example 2(e)

Find the values of *x* for which the function is continuous:

$$G(x)=\begin{cases}\dfrac{1}{x} & \textbf{if } x>0\\ -1 & \textbf{if } x\le 0\end{cases}$$

Solution:

The function *G* is discontinuous at *x* = 0 because the limit of *G* fails to exist as *x* approaches 0. It is continuous everywhere else.



Properties of Continuous Functions

1. The constant function $f(x) = c$ is continuous everywhere.
2. The identity function $f(x) = x$ is continuous everywhere.

If f and g are continuous at $x = a$, then

3. $[f(x)]^n$, where n is a real number, is continuous at $x = a$ whenever it is defined at that number.
4. $f \pm g$ is continuous at $x = a$.
5. fg is continuous at $x = a$.
6. f/g is continuous at $g(a) \neq 0$.

Properties of Continuous Functions

Using these properties, we can obtain the following additional properties.

1. A polynomial function $y = P(x)$ is continuous at every value of x.

2. A rational function $R(x) = p(x)/q(x)$ is continuous at every value of x where $q(x) \neq 0$.

Example 3(a)

Find the values of *x* for which the function is continuous.

$$f(x) = 3x^3 + 2x^2 - x + 10$$

Solution:

The function *f* is a polynomial function of degree 3, so *f*(*x*) is continuous for all values of *x*.

Example 3(b)

Find the values of *x* for which the function is continuous.

$$g(x) = \frac{8x^{10} - 4x^2 + 1}{x^2 + 1}$$

Solution:

The function *g* is a rational function.

Observe that the denominator of *g* is never equal to zero.

Therefore, we conclude that *g*(*x*) is continuous for all values of *x*.

Example 3(c)

Find the values of x for which the function is continuous.

$$h(x) = \frac{4x^3 - 3x^2 + 1}{x^2 - 3x + 2}$$

Solution:

The function h is a rational function.

In this case, however, the denominator of h is equal to zero at $x = 1$ and $x = 2$, which we can see by factoring.

Therefore, we conclude that $h(x)$ is continuous everywhere except at $x = 1$ and $x = 2$.

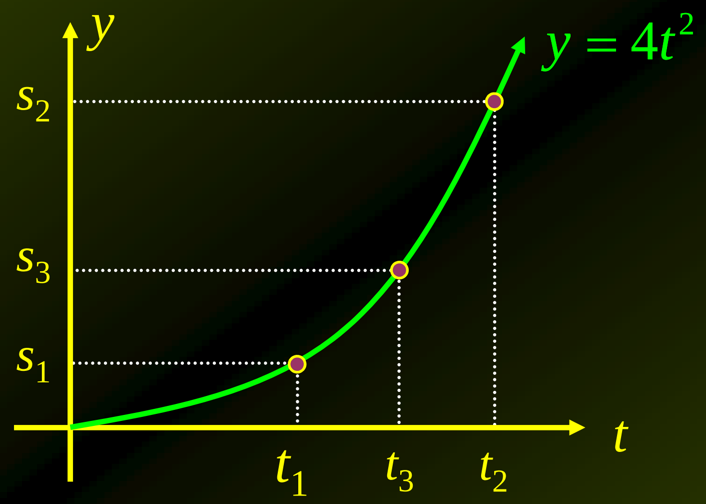
Intermediate Value Theorem

Let's look again at the maglev example.

The train cannot vanish at any instant of time and cannot skip portions of track and reappear elsewhere.

Intermediate Value Theorem

Mathematically, recall that the position of the maglev is a function of time given by $f(t) = 4t^2$ for $0 \leq t \leq 30$:

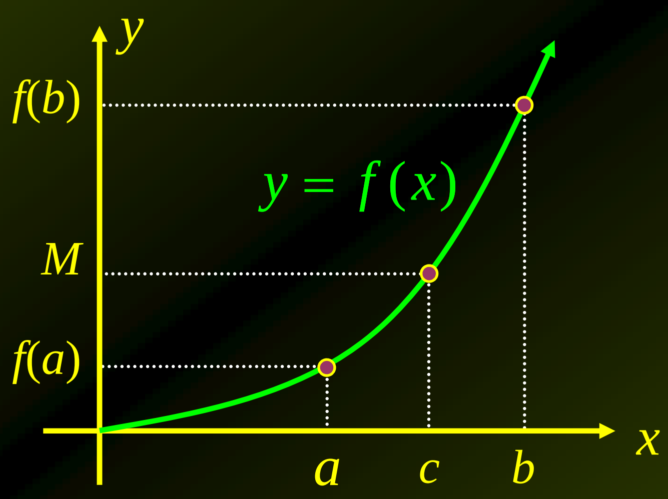


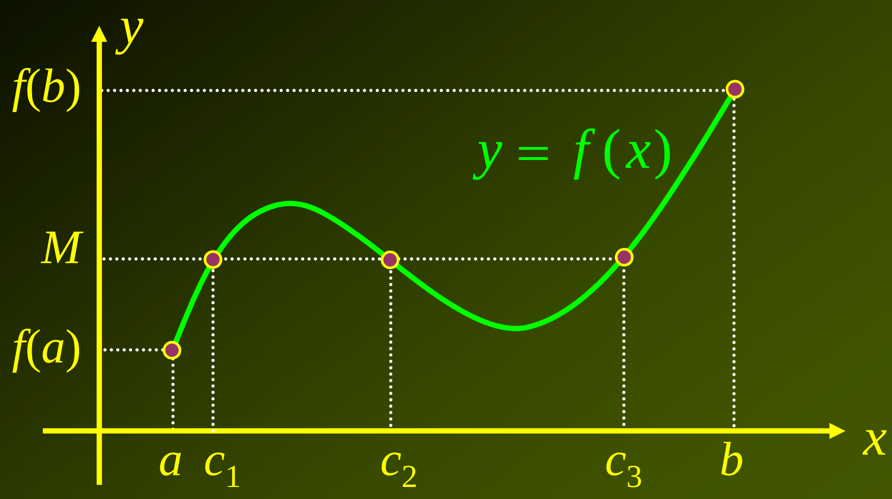
Suppose the position of the maglev is s_1 at some time t_1 and its position is s_2 at some time t_2. Then, if s_3 is any number between s_1 and s_2, there must be at least one t_3 between t_1 and t_2 giving the time at which the maglev is at s_3 ($f(t_3) = s_3$).

Theorem 4: Intermediate Value Theorem

The Maglev example carries the gist of the intermediate value theorem:

If f is a continuous function on a closed interval $[a, b]$ and M is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = M$.

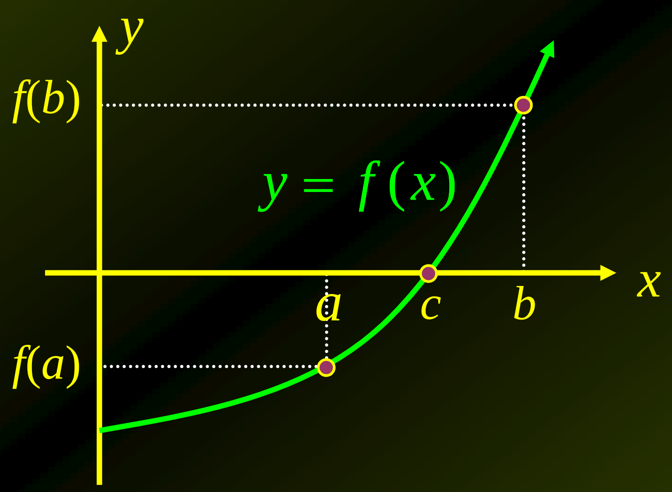


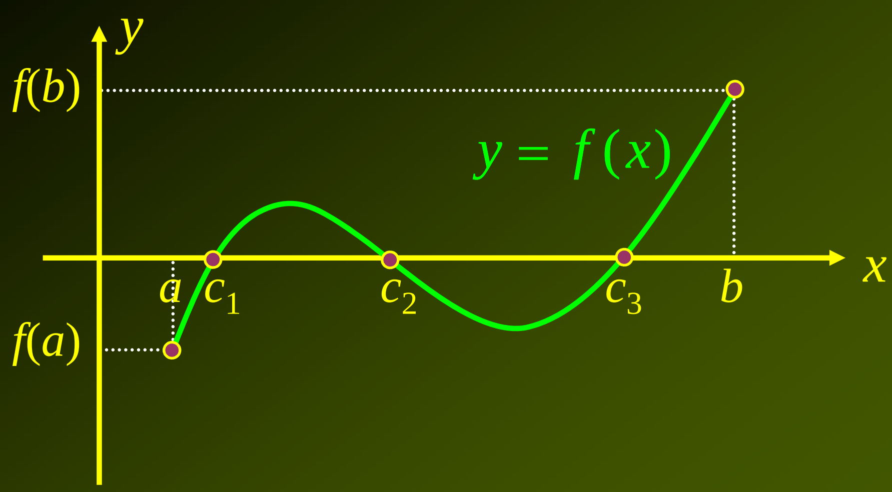


Theorem 5: Existence of Zeros of a Continuous Function

A special case of this theorem is when a continuous function crosses the *x* axis.

If *f* is a continuous function on a closed interval [*a*, *b*], and if *f*(*a*) and *f*(*b*) have opposite signs, then there is at least one solution of the equation *f*(*x*) = 0 in the interval (*a*, *b*).





Example 5

Let $f(x) = x^3 + x + 1$.

a. Show that f is continuous for all values of x.

b. Compute $f(-1)$ and $f(1)$ and use the results to deduce that there must be at least one number $x = c$, where c lies in the interval $(-1, 1)$ and $f(c) = 0$.

Example 5 – *Solution*

a. The function f is a polynomial function of degree 3 and is therefore continuous everywhere.

b. $f(-1) = (-1)^3 + (-1) + 1 = -1$ and $f(1) = (1)^3 + (1) + 1 = 3$

Since $f(-1)$ and $f(1)$ have opposite signs, Theorem 5 tells us that there must be at least one number $x = c$ with $-1 < c < 1$ such that $f(c) = 0$.