2 FUNCTIONS, LIMITS, AND THE DERIVATIVE

2.6 The Derivative

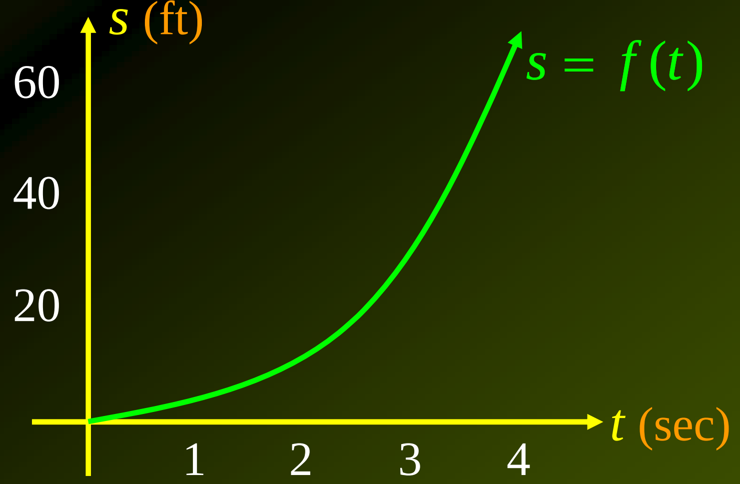
An Intuitive Example

Consider the maglev example from Section 2.4.

The position of the maglev is a function of time given by

$$s = f(t) = 4t^2 \qquad (0 \leq t \leq 30)$$

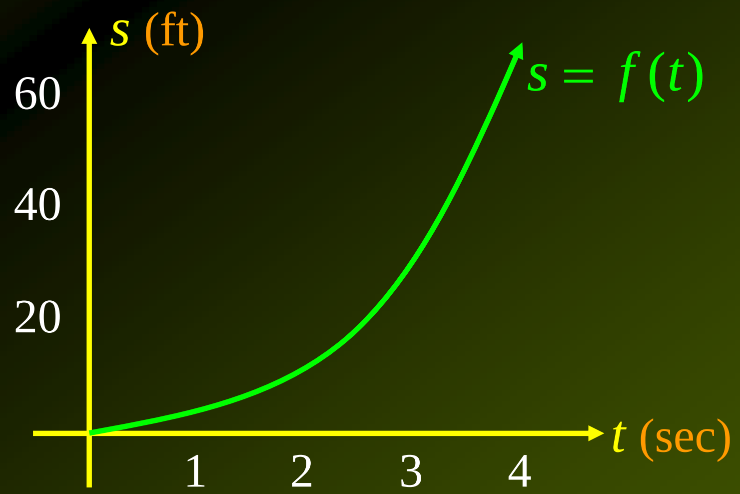
where s is measured in feet and t in seconds. Its graph is:



An Intuitive Example

The graph rises slowly at first but more rapidly over time. This suggests the steepness of $f(t)$ is related to the *speed* of the maglev, which also increases over time.

If so, we might be able to find the speed of the maglev at any given time by finding the steepness of f at that time. But how do we find the steepness of a point in a curve?

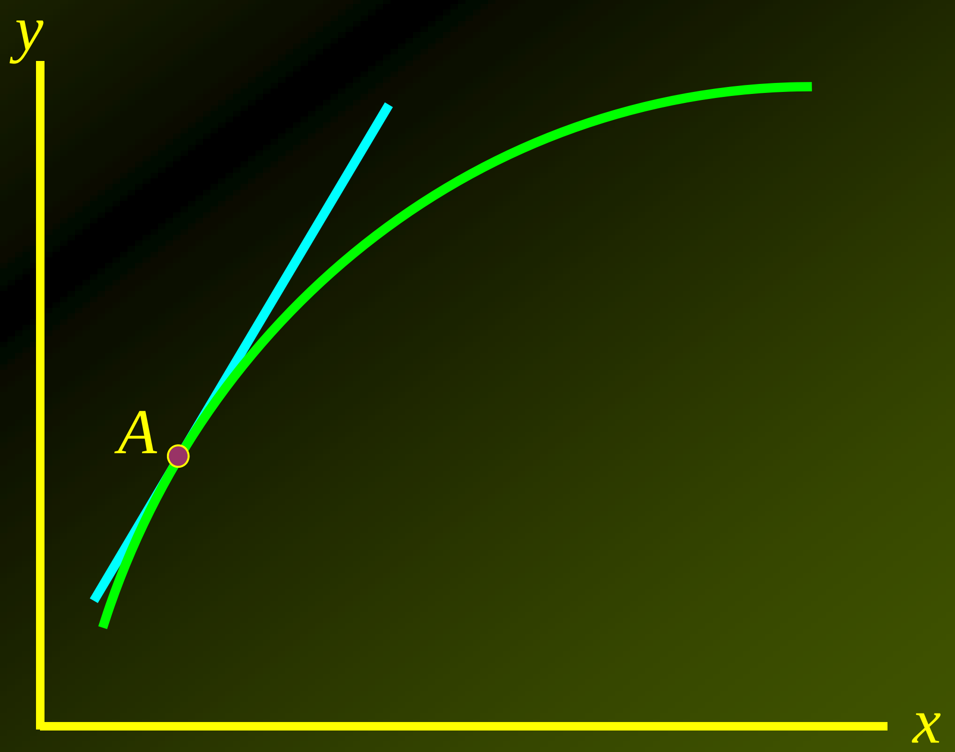


Slopes of Lines and of Curves

The slope at a point of a curve is given by the slope of the tangent to the curve at that point:

Suppose we want to find the slope at point *A*.

The tangent line has the same slope as the curve does at point *A*.

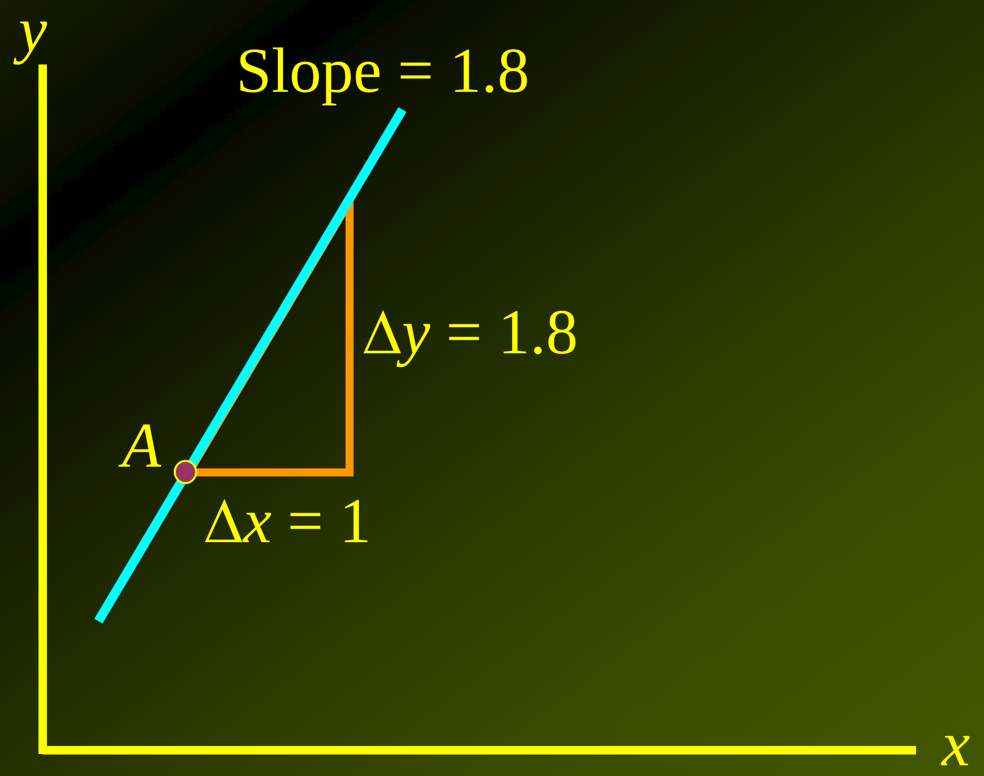


Slopes of Lines and of Curves

The slope of a point in a curve is given by the slope of the tangent to the curve at that point:

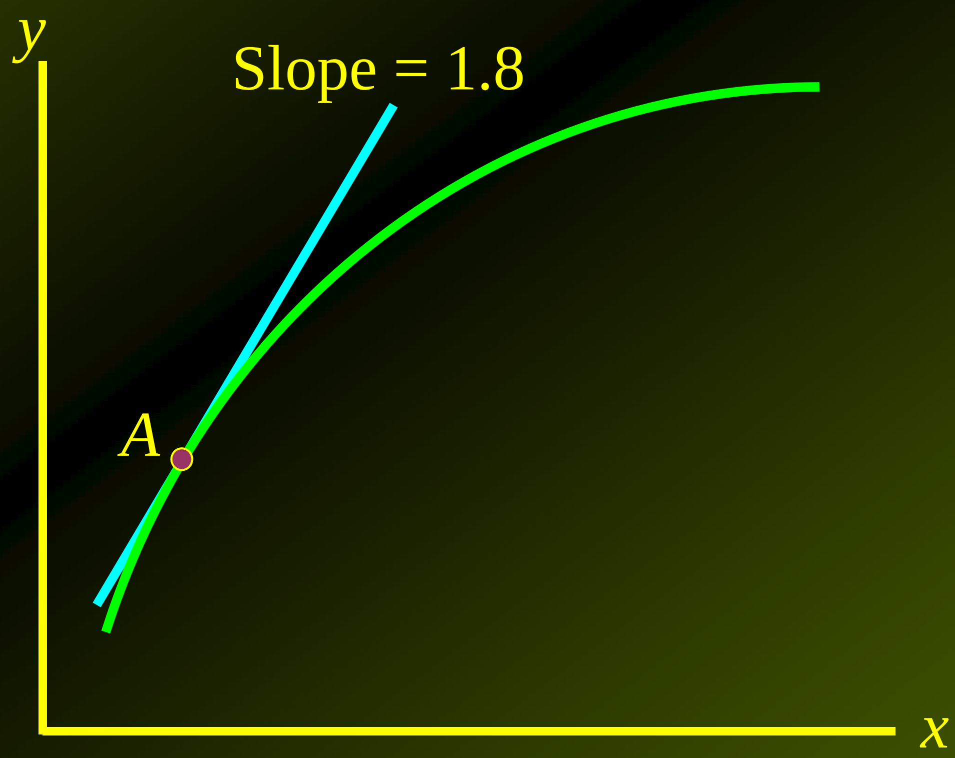
The slope of the tangent in this case is 1.8:

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{1.8}{1} = 1.8$$



Slopes of Lines and of Curves

The slope at a point of a curve is given by the slope of the tangent to the curve at that point:

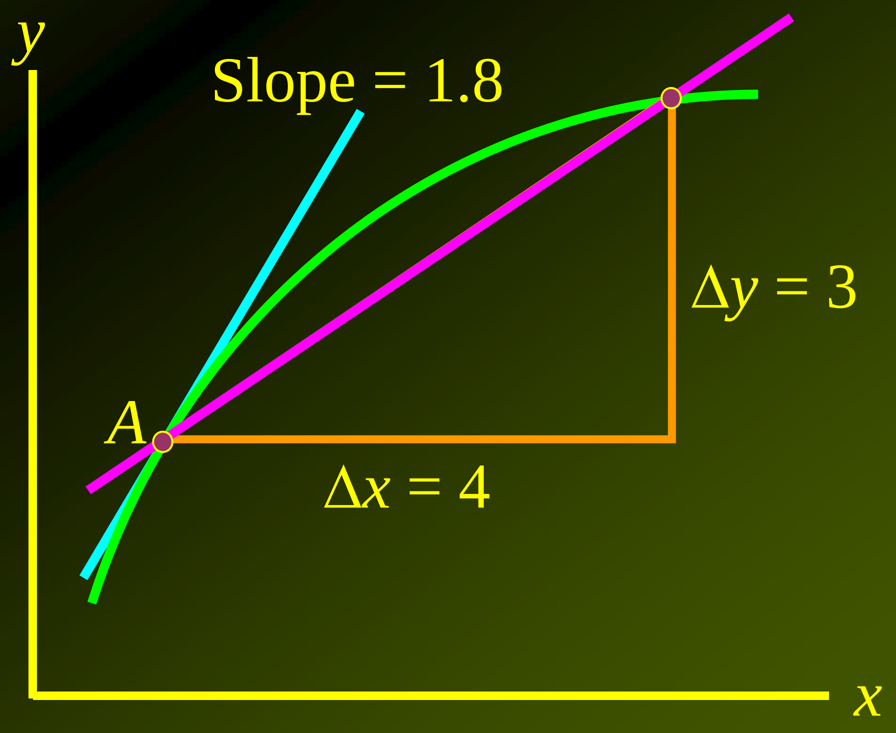


Slopes of Lines and of Curves

To calculate accurately the slope of a tangent to a curve, we must make the change in *x* as small as possible:

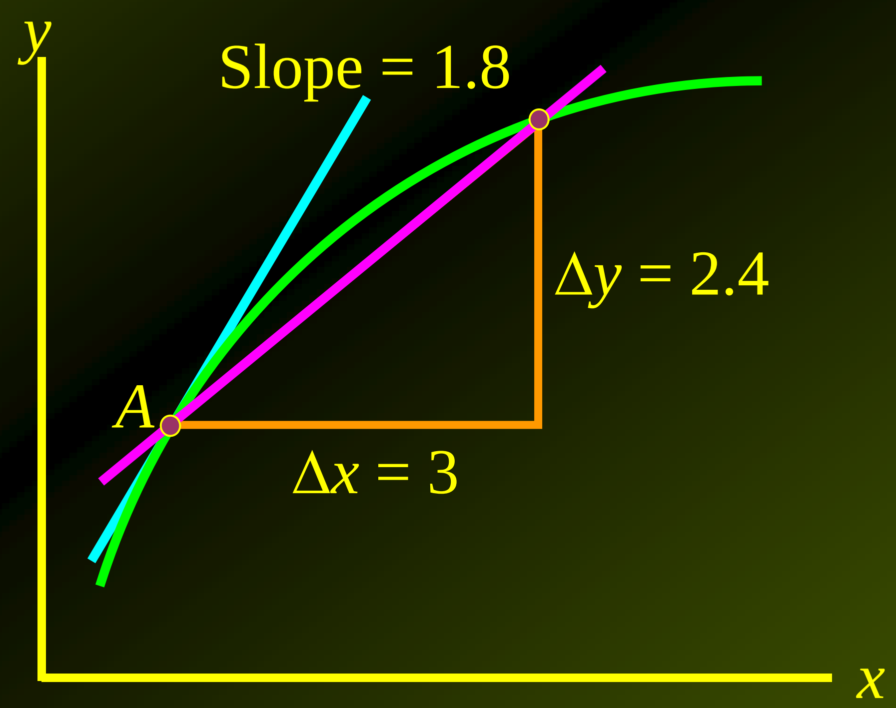
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{3}{4} = 0.75$$

As we let Δx get smaller, the slope of the secant becomes more and more similar to the slope of the tangent to the curve at that point.



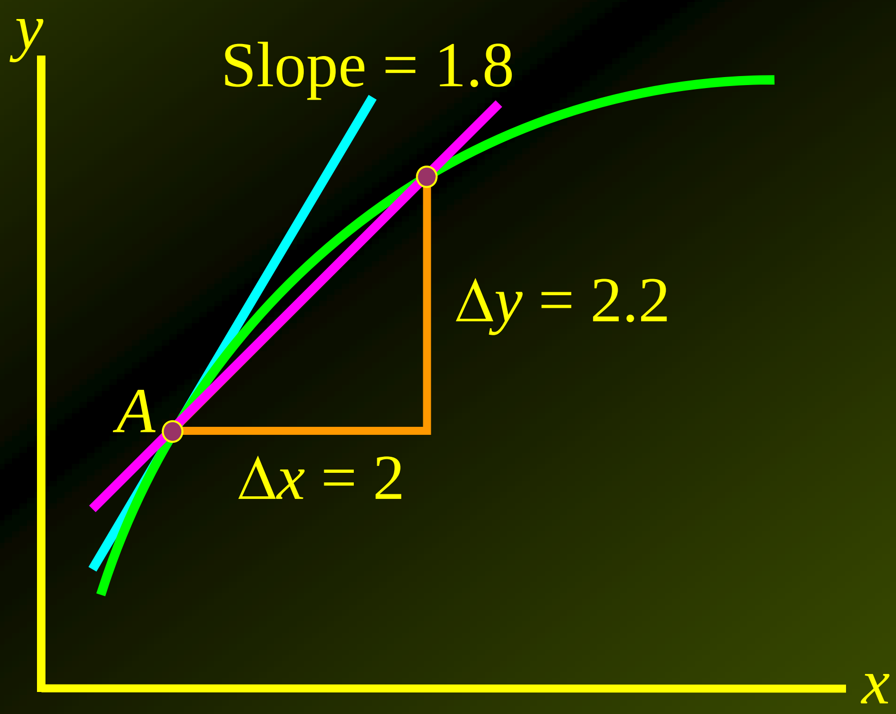
Slopes of Lines and of Curves

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{2.4}{3} = 0.8$$



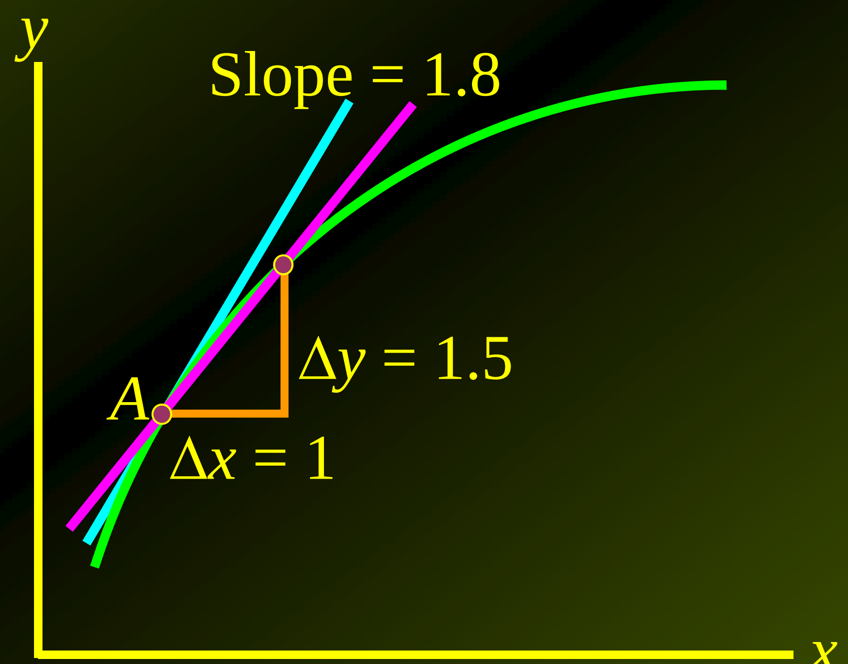
Slopes of Lines and of Curves

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{2.2}{2} = 1.1$$



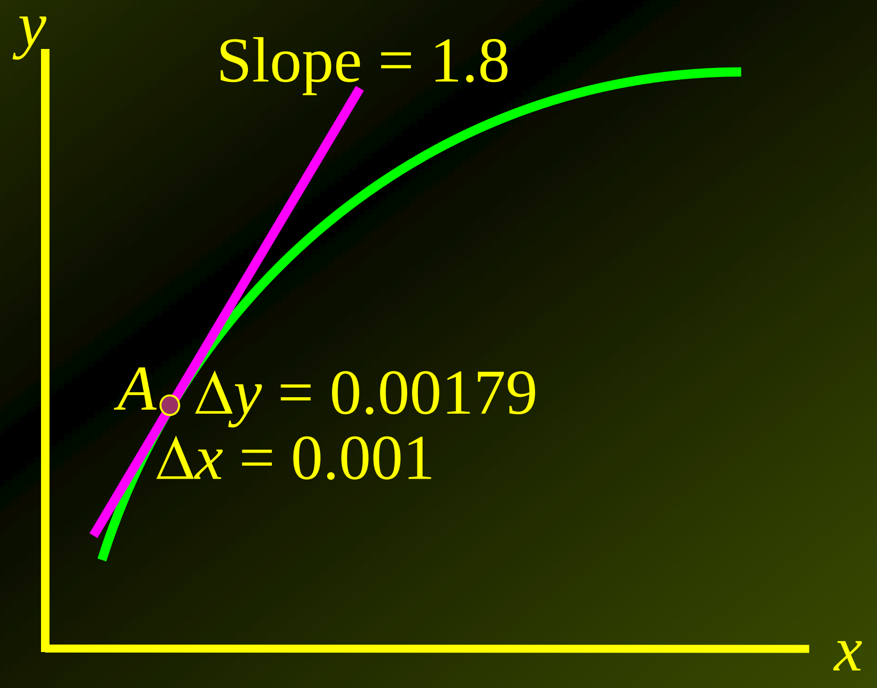
Slopes of Lines and of Curves

$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{1.5}{1} = 1.5$$



Slopes of Lines and of Curves

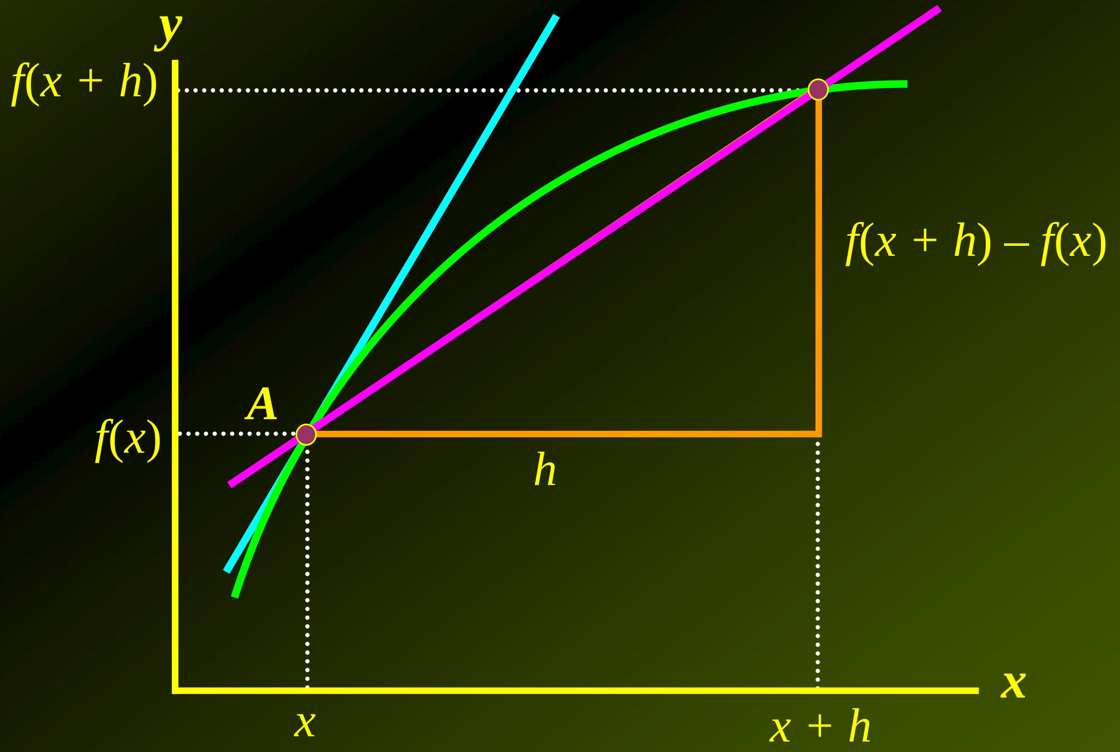
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{0.00179}{0.001} \approx 1.8$$



Slopes of Lines and of Curves

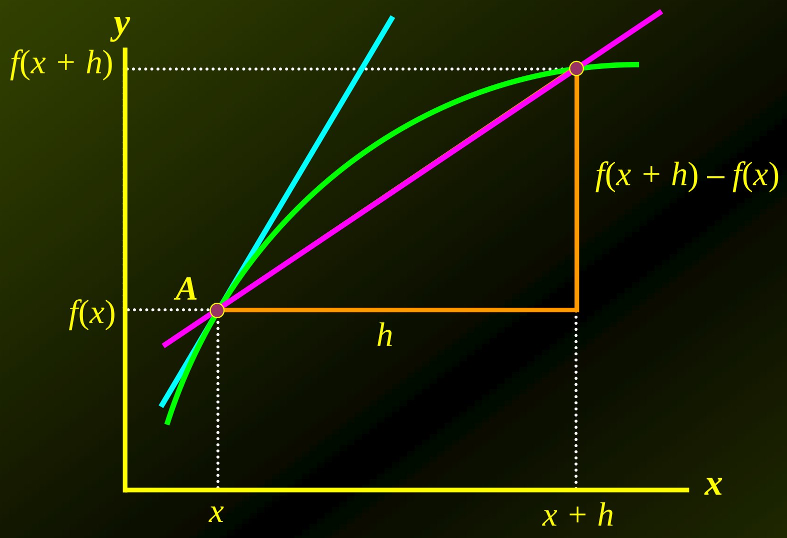
In general, we can express the slope of the secant as follows:

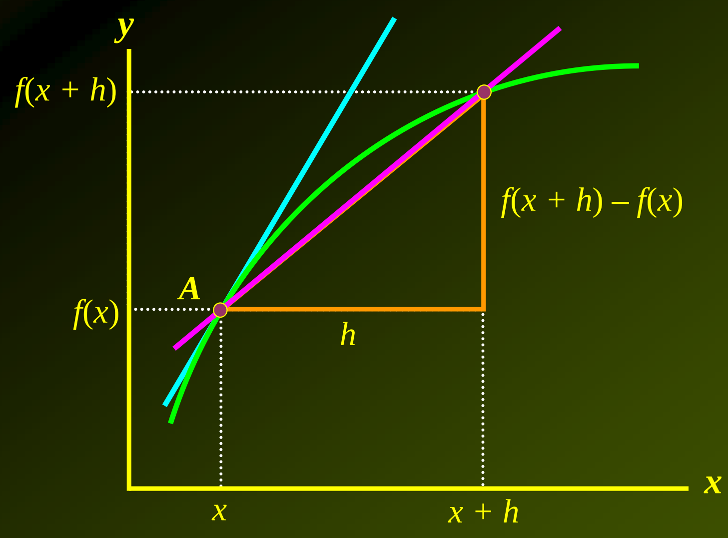
$$\text{Slope} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$



Slopes of Lines and of Curves

Thus, as *h* approaches zero, the slope of the secant approaches the slope of the tangent to the curve at that point:





Slopes of Lines and of Curves

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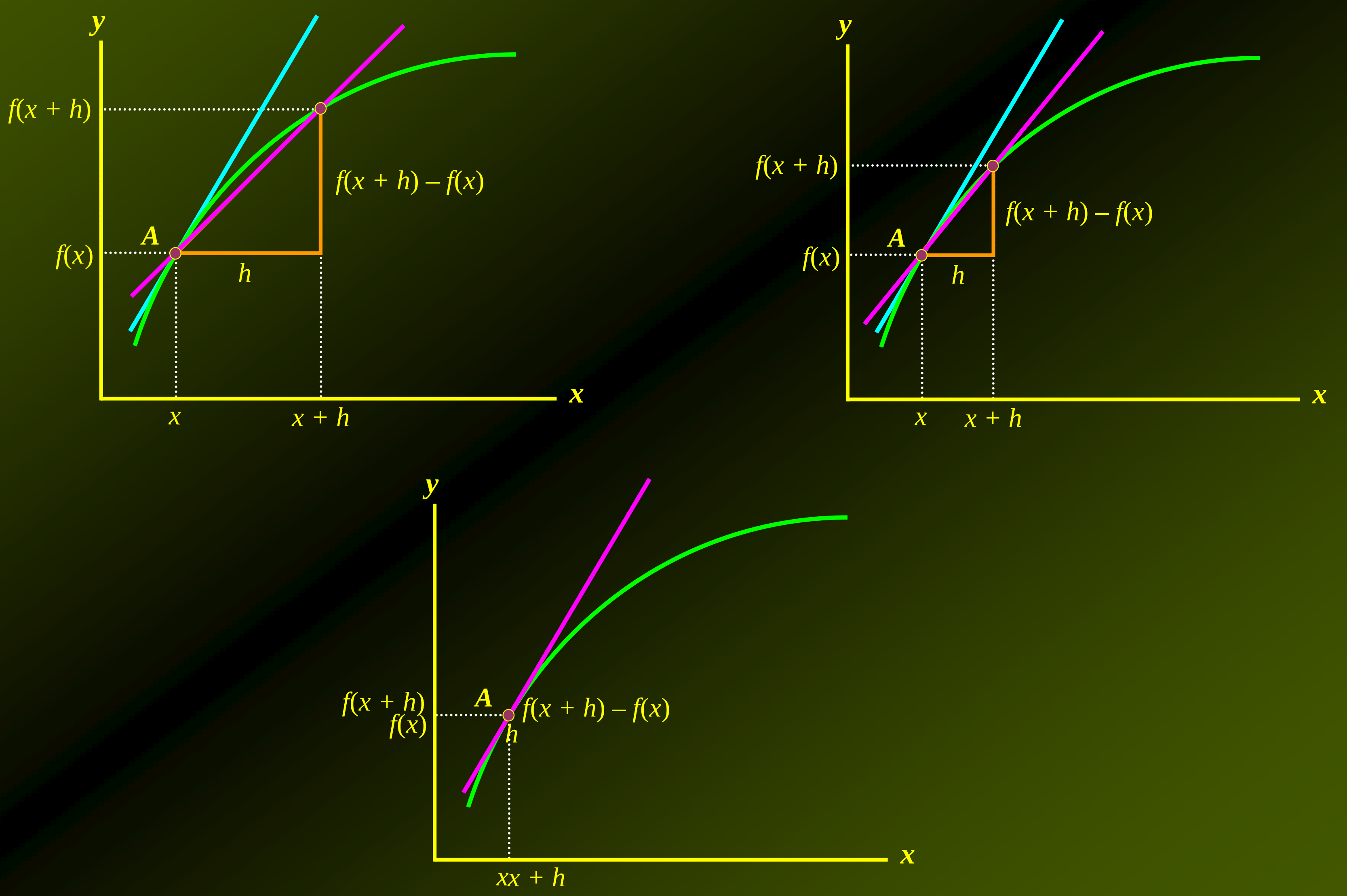


Slopes of Lines and of Curves

Thus, as *h* approaches zero, the slope of the secant approaches the slope of the tangent to the curve at that point.

Expressed in limits notation:

The slope of the tangent line to the graph of *f* at the point *P*(*x*, *f*(*x*)) is given by

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if it exists.

Average Rates of Change

We can see that measuring the slope of the tangent line to a graph is mathematically equivalent to finding the rate of change of *f* at *x*.

The number $f(x + h) - f(x)$ measures the change in *y* that corresponds to a change *h* in *x*.

Then the difference quotient $\frac{f(x+h)-f(x)}{h}$

measures the average rate of change of *y* with respect to *x* over the interval $[x, x + h]$.

In the maglev example, if *y* measures the position the train at time *x*, then the quotient give the average velocity of the train over the time interval $[x, x + h]$.

Average Rates of Change

The average rate of change of *f* over the interval $[x, x + h]$ or slope of the secant line to the graph of *f* through the points $(x, f(x))$ and $(x + h, f(x + h))$ is

$$\frac{f(x+h) - f(x)}{h}$$

Instantaneous Rates of Change

By taking the limit of the difference quotient as *h* goes to zero, evaluating

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

we obtain the rate of change of *f* at *x*.

This is known as the instantaneous rate of change of *f* at *x* (as opposed to the average rate of change).

In the maglev example, if *y* measures the position of a train at time *x*, then the limit gives the velocity of the train at time *x*.

Instantaneous Rates of Change

The instantaneous rate of change of f at x or slope of the tangent line to the graph of f at $(x, f(x))$ is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This limit is called the derivative of f at x .

The Derivative of a Function

The derivative of a function *f* with respect to *x* is the function f' (read "*f* prime").

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of f' is the set of all *x* where the limit exists.

Thus, the derivative of function *f* is a function f' that gives the slope of the tangent to the line to the graph of *f* at any point $(x, f(x))$ and also the rate of change of *f* at *x*.

The Derivative of a Function

Four Step Process for Finding $f'(x)$

1. Compute $f(x + h)$.

2. Form the difference $f(x + h) - f(x)$.

3. Form the quotient $\dfrac{f(x+h)-f(x)}{h}$.

4. Compute $f'(x)=\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

Example 2

Find the slope of the tangent line to the graph $f(x) = 3x + 5$ at any point $(x, f(x))$.

Solution:

The required slope is given by the derivative of f at x. To find the derivative, we use the four-step process:

Step 1. $f(x + h) = 3(x + h) + 5 = 3x + 3h + 5$.

Step 2. $f(x + h) - f(x) = 3x + 3h + 5 - (3x + 5) = 3h$.

Step 3. $$\frac{f(x+h) - f(x)}{h} = \frac{3h}{h} = 3.$$

Step 4. $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 3 = 3.$$

Example 3(a)

Find the slope of the tangent line to the graph $f(x) = x^2$ at any point $(x, f(x))$.

Solution:

The required slope is given by the derivative of f at x. To find the derivative, we use the four-step process:

Step 1. $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$.

Step 2. $f(x + h) - f(x) = x^2 + 2xh + h^2 - x^2 = h(2x + h)$.

Step 3. $$\frac{f(x+h)-f(x)}{h} = \frac{h(2x+h)}{h} = 2x+h.$$

Step 4. $$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h) = 2x.$$

Example 3(b)

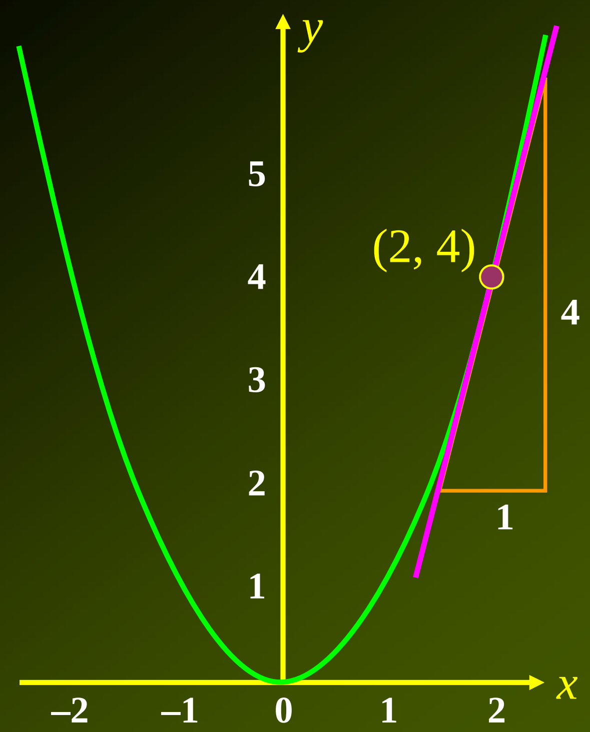
Find the slope of the tangent line to the graph $f(x) = x^2$ at any point $(x, f(x))$. The slope of the tangent line is given by $f'(x) = 2x$. Now, find and interpret $f'(2)$.

Solution:

$$f'(2) = 2(2) = 4.$$

This means that, at the point (2, 4)…

… the slope of the tangent line to the graph is 4.



Applied Example 7 – *Demand for Tires*

The management of Titan Tire Company has determined that the weekly demand function of their Super Titan tires is given by

$$p = f(x) = 144 - x^2$$

where p is measured in dollars and x is measured in thousands of tires.

Find the average rate of change in the unit price of a tire if the quantity demanded is between 5000 and 6000 tires; between 5000 and 5100 tires; and between 5000 and 5010 tires.

What is the instantaneous rate of change of the unit price when the quantity demanded is 5000 tires?

Applied Example 7 – *Solution*

The average rate of change of the unit price of a tire if the quantity demanded is between x and $x + h$ is

$$\frac{f(x+h)-f(x)}{h}=\frac{[144-(x+h)^2]-(144-x^2)}{h}$$

$$=\frac{144-x^2-2xh-h^2-144+x^2}{h}$$

$$=\frac{-2xh-h^2}{h}=\frac{h(-2x-h)}{h}$$

$$=-2x-h$$

Applied Example 7 – *Solution*

cont'd

The average rate of change is given by $–2x – h$.

To find the average rate of change of the unit price of a tire when the quantity demanded is between 5000 and 6000 tires [5, 6], we take $x = 5$ and $h = 1$, obtaining

$$-2(5)-1=-11$$

or –$11 per 1000 tires.

Similarly, with $x = 5$, and $h = 0.1$, we obtain

$$-2(5)-0.1=-10.1$$

or –$10.10 per 1000 tires.

Applied Example 7 – *Solution*

cont'd

Finally, with $x = 5$, and $h = 0.01$, we get

$$-2(5) - 0.01 = -10.01$$

or –\$10.01 per 1000 tires.

The instantaneous rate of change of the unit price of a tire when the quantity demanded is x tires is given by

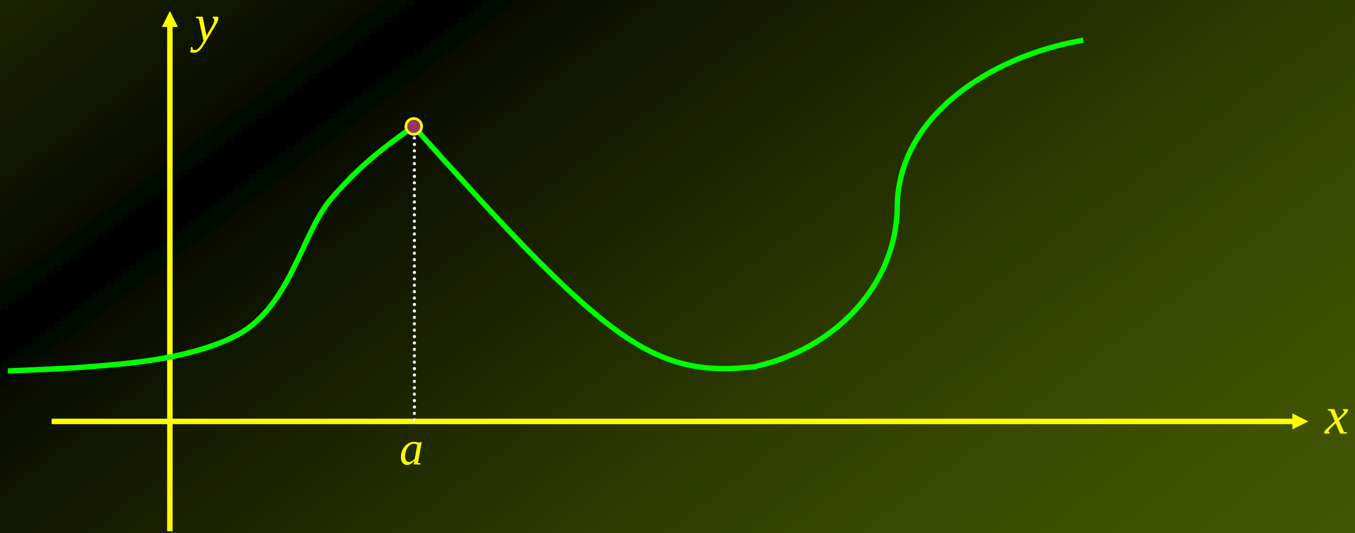
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (-2x - h) = -2x$$

In particular, the instantaneous rate of change of the price per tire when quantity demanded is 5000 is given by –2(5), or –\$10 per tire.

Differentiability and Continuity

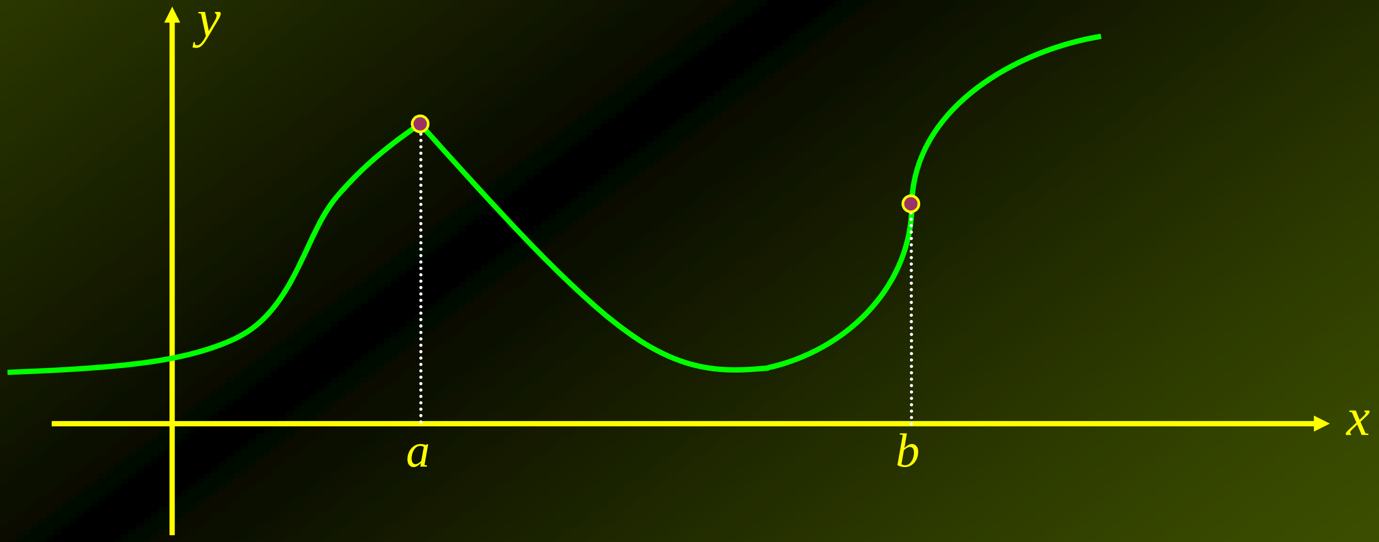
Sometimes, one encounters continuous functions that fail to be differentiable at certain values in the domain of the function f. *For example*, consider the continuous function f below:

1. It fails to be differentiable at $x = a$, because the graph makes an abrupt change (a corner) at that point. (It is not clear what the slope is at that point)



Differentiability and Continuity

2. It also fails to be differentiable and $x = b$ because the slope is not defined at that point.



Applied Example 8 – *Wages*

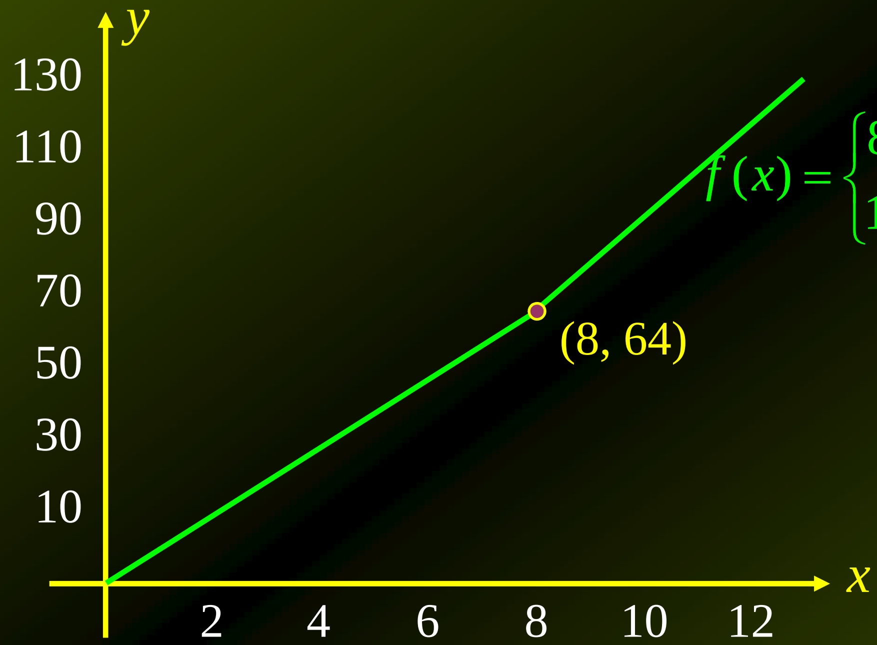
Mary works at the B&O department store, where, on a weekday, she is paid $8 an hour for the first 8 hours and $12 an hour of overtime.

The function
$$f(x) = \begin{cases} 8x & \textbf{if } 0 \le x \le 8 \\ 12x - 32 & \textbf{if } x > 8 \end{cases}$$

gives Mary's earnings on a weekday in which she worked *x* hours.

Sketch the graph of the function *f* and explain why it is not differentiable at $x = 8$.

Applied Example 8 – *Solution*



$$f(x) = \begin{cases} 8x & \textbf{if } 0 \le x \le 8 \\ 12x - 32 & \textbf{if } x > 8 \end{cases}$$

The graph of f has a corner at $x = 8$ and so is not differentiable at that point.