3 DIFFERENTIATION

3.1 Basic Rules of Differentiation

Four Basic Rules

We've learned that to find the rule for the derivative f' of a function f, we first find the difference quotient

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But this method is tedious and time consuming, even for relatively simple functions.

This chapter we will develop rules that will simplify the process of finding the derivative of a function.

Rule 1: Derivative of a Constant

We will use the notation $\frac{d}{dx}[f(x)]$

To mean "the derivative of *f* with respect to *x* at *x*."

Rule 1: Derivative of a constant

$$\frac{d}{dx}\left(c\right)=0$$

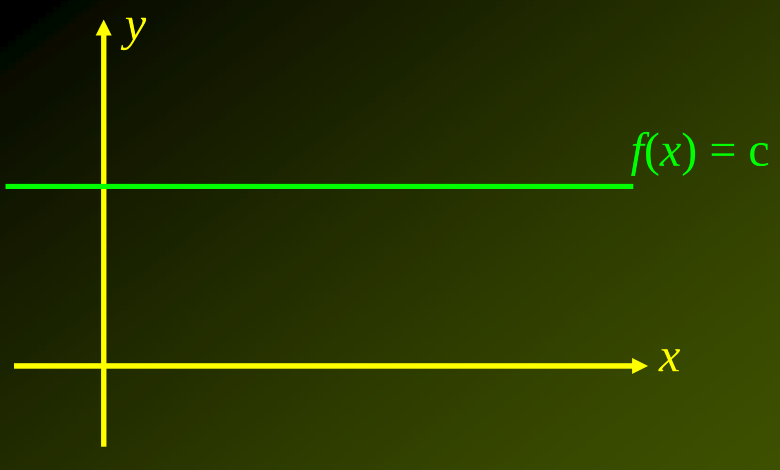
The derivative of a constant function is equal to zero.

Rule 1: Derivative of a Constant

We can see geometrically why the derivative of a constant must be zero.

The graph of a constant function is a straight line parallel to the *x* axis.

Such a line has a slope that is constant with a value of zero. Thus, the derivative of a constant must be zero as well.



Rule 1: Derivative of a Constant

We can use the definition of the derivative to demonstrate this:

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{c - c}{h} \\
&= \lim_{h \to 0} 0 \\
&= 0
\end{aligned}
$$

Rule 2: The Power Rule

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If *n* is any real number, then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Rule 2: The Power Rule

Lets verify this rule for the special case of $n = 2$.

If $f(x) = x^2$, then

$$f'(x) = \frac{d}{dx}\left(x^2\right) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x$$

Example 2

If $f(x) = x$, then $f'(x) = \frac{d}{dx}\left(x\right) = 1 \cdot x^{1-1} = x^0 = 1$

If $f(x) = x^8$, then $f'(x) = \frac{d}{dx}\left(x^8\right) = 8 \cdot x^{8-1} = 8x^7$

If $f(x) = x^{5/2}$, then $f'(x) = \frac{d}{dx}\left(x^{5/2}\right) = \frac{5}{2} \cdot x^{5/2-1} = \frac{5}{2}x^{3/2}$

Example 3(a)

Find the derivative of $f(x) = \sqrt{x}$

Solution:

$$f'(x) = \frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right)$$

$$= \frac{1}{2}x^{1/2-1}$$

$$= \frac{1}{2}x^{-1/2}$$

$$= \frac{1}{2\sqrt{x}}$$

Example 3(b)

Find the derivative of $f(x) = \dfrac{1}{\sqrt[3]{x}}$

Solution:

$$f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt[3]{x}}\right) = \frac{d}{dx}\left(x^{-1/3}\right)$$

$$= -\frac{1}{3}x^{-1/3-1}$$

$$= -\frac{1}{3}x^{-4/3}$$

$$= -\frac{1}{3x^{4/3}}$$

Rule 3: Derivative of a Constant Multiple Function

Rule 3: Derivative of a Constant Multiple Function

If c is any constant real number, then

$$\frac{d}{dx}\left[cf(x)\right] = c\frac{d}{dx}\left[f(x)\right]$$

Example 4(a)

Find the derivative of $f(x) = 5x^3$

Solution:

$$f'(x) = \frac{d}{dx}\left(5x^3\right)$$

$$= 5\frac{d}{dx}\left(x^3\right)$$

$$= 5\left(3x^2\right)$$

$$= 15x^2$$

Example 4(b)

Find the derivative of $f(x) = \dfrac{3}{\sqrt{x}}$

Solution:

$$f'(x) = \frac{d}{dx}\left(3x^{-1/2}\right)$$

$$= 3\left(-\frac{1}{2}x^{-3/2}\right)$$

$$= -\frac{3}{2x^{3/2}}$$

Rule 4: The Sum Rule

Rule 4: The Sum Rule

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Example 5(a)

Find the derivative of $f(x) = 4x^5 + 3x^4 - 8x^2 + x + 3$

Solution:

$$f'(x) = \frac{d}{dx}\left(4x^5 + 3x^4 - 8x^2 + x + 3\right)$$

$$= 4\frac{d}{dx}\left(x^5\right) + 3\frac{d}{dx}\left(x^4\right) - 8\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(x) + \frac{d}{dx}(3)$$

$$= 4\left(5x^4\right) + 3\left(4x^3\right) - 8\left(2x\right) + 1 + 0$$

$$= 20x^4 + 12x^3 - 16x + 1$$

Example 5(b)

Find the derivative of $g(t) = \frac{t^2}{5} + \frac{5}{t^3}$

Solution:

$$g'(t) = \frac{d}{dt}\left(\frac{t^2}{5} + \frac{5}{t^3}\right)$$

$$= \frac{d}{dt}\left(\frac{1}{5}t^2 + 5t^{-3}\right)$$

$$= \frac{1}{5} \cdot \frac{d}{dt}\left(t^2\right) + 5\frac{d}{dt}\left(t^{-3}\right)$$

$$= \frac{2t}{5} - \frac{15}{t^4} = \frac{2t^5 - 75}{5t^4}$$

Applied Example 7 – *Conservation of a Species*

A group of marine biologists at the Neptune Institute of Oceanography recommended that a series of conservation measures be carried out over the next decade to save a certain species of whale from extinction. After implementing the conservation measure, the population of this species is expected to be

$$N(t) = 3t^3 + 2t^2 - 10t + 600 \qquad (0 \le t \le 10)$$

where $N(t)$ denotes the population at the end of year t. Find the rate of growth of the whale population when $t = 2$ and $t = 6$. How large will the whale population be 8 years after implementing the conservation measures?

Applied Example 7 – *Solution*

The rate of growth of the whale population at any time *t* is given by

$$N'(t) = 9t^2 + 4t - 10$$

In particular, for *t* = 2, we have

$$N'(2) = 9(2)^2 + 4(2) - 10 = 34$$

And for *t* = 6, we have

$$N'(6) = 9(6)^2 + 4(6) - 10 = 338$$

Thus, the whale population's rate of growth will be 34 whales per year after 2 years and 338 per year after 6 years.

Applied Example 7 – *Solution*

cont'd

The whale population at the end of the eighth year will be

$$N(8) = 3(8)^3 + 2(8)^2 - 10(8) + 600$$

$$= 2184 \text{ whales}$$