3 DIFFERENTIATION

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3.2 The Product and Quotient Rules

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Rule 5: The Product Rule

The derivative of the product of two differentiable functions is given by

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$$

Example 1

Find the derivative of $f(x) = \left(2x^2 - 1\right)\left(x^3 + 3\right)$

Solution:

$$f'(x) = \left(2x^2 - 1\right)\frac{d}{dx}\left(x^3 + 3\right) + \left(x^3 + 3\right)\frac{d}{dx}\left(2x^2 - 1\right)$$

$$= \left(2x^2 - 1\right)\left(3x^2\right) + \left(x^3 + 3\right)\left(4x\right)$$

$$= 6x^4 - 3x^2 + 4x^4 + 12x$$

$$= x\left(10x^3 - 3x + 12\right)$$

Example 2

Find the derivative of $f(x) = x^3\left(\sqrt{x}+1\right)$

Solution:

$$f'(x) = x^3 \frac{d}{dx}\left(x^{1/2}+1\right) + \left(x^{1/2}+1\right)\frac{d}{dx}x^3$$

$$= x^3\left(\frac{1}{2}x^{-1/2}\right) + \left(x^{1/2}+1\right)3x^2$$

$$= \frac{1}{2}x^{5/2} + 3x^{5/2} + 3x^2$$

$$= \frac{7}{2}x^{5/2} + 3x^2$$

Rule 6: The Quotient Rule

The derivative of the quotient of two differentiable functions is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)\right]^2} \qquad \left(g(x) \neq 0\right)$$

Example 3

Find the derivative of $f(x) = \dfrac{x}{2x-4}$

Solution:

$$f'(x) = \frac{(2x-4)\frac{d}{dx}(x) - x\frac{d}{dx}(2x-4)}{(2x-4)^2}$$

$$= \frac{(2x-4)(1) - x(2)}{(2x-4)^2}$$

$$= \frac{2x-4-2x}{(2x-4)^2} = -\frac{4}{(2x-4)^2}$$

Example 4

Find the derivative of $f(x) = \dfrac{x^2+1}{x^2-1}$

Solution:

$$f'(x) = \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$= \frac{\left(x^2-1\right)(2x) - \left(x^2+1\right)(2x)}{\left(x^2-1\right)^2}$$

$$= \frac{2x^3-2x-2x^3-2x}{\left(x^2-1\right)^2} = -\frac{4x}{\left(x^2-1\right)^2}$$

Applied Example 6 – *Rate of Change of DVD Sales*

The sales (in millions of dollars) of DVDs of a hit movie *t* years from the date of release is given by

$$S(t) = \frac{5t}{t^2 + 1}$$

Find the rate at which the sales are changing at time *t*.

How fast are the sales changing at:

a. The time the DVDs are released ($t = 0$)?

b. And two years from the date of release ($t = 2$)?

Applied Example 6 – *Solution*

The rate of change at which the sales are changing at time *t* is given by

$$S'(t) = \frac{d}{dt}\left[\frac{5t}{t^2+1}\right]$$

$$= \frac{\left(t^2+1\right)(5)-(5t)(2t)}{\left(t^2+1\right)^2}$$

$$= \frac{5t^2+5-10t^2}{\left(t^2+1\right)^2}$$

$$= \frac{5\left(1-t^2\right)}{\left(t^2+1\right)^2}$$

Applied Example 6 – *Solution*

cont'd

a. The rate of change at which the sales are changing when the DVDs are released ($t = 0$) is

$$S'(0) = \frac{5\left[1 - (0)^2\right]}{\left[(0)^2 + 1\right]^2}$$

$$= \frac{5(1)}{(1)^2} = 5$$

That is, sales are increasing by \$5 million per year.

Applied Example 6 – *Solution*

cont'd

b. The rate of change two years after the DVDs are released ($t = 2$) is

$$S'(2) = \frac{5\left[1-(2)^2\right]}{\left[(2)^2+1\right]^2}$$

$$= \frac{5(1-4)}{(4+1)^2}$$

$$= -\frac{15}{25} = -\frac{3}{5} = -0.6$$

That is, sales are decreasing by $600,000 per year.