3 DIFFERENTIATION

3.3 The Chain Rule

Deriving Composite Functions

Consider the function $h(x) = \left(x^2 + x + 1\right)^2$

To compute $h'(x)$, we can first expand $h(x)$

$$h(x) = \left(x^2 + x + 1\right)^2 = \left(x^2 + x + 1\right)\left(x^2 + x + 1\right)$$
$$= x^4 + 2x^3 + 3x^2 + 2x + 1$$

and then derive the resulting polynomial

$$h'(x) = 4x^3 + 6x^2 + 6x + 2$$

But how should we derive a function like $H(x)$?

$$H(x) = \left(x^2 + x + 1\right)^{100}$$

Deriving Composite Functions

Note that $H(x)=\left(x^2+x+1\right)^{100}$ is a composite function:

H(*x*) is composed of two simpler functions

$$f(x)=x^2+x+1 \qquad \textbf{and} \qquad g(x)=x^{100}$$

So that

$$H(x)=g\left[f(x)\right]=\left[f(x)\right]^{100}=\left(x^2+x+1\right)^{100}$$

We can use this to find the derivative of *H*(*x*).

Deriving Composite Functions

To find the derivative of the composite function $H(x)$:

We let $u = f(x) = x^2 + x + 1$ and $y = g(u) = u^{100}$.

Then we find the derivatives of each of these functions

$$\frac{du}{dx} = f'(x) = 2x + 1 \quad \text{and} \quad \frac{dy}{du} = g'(u) = 100u^{99}$$

The ratios of these derivatives suggest that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 100u^{99}(2x+1)$$

Substituting $x^2 + x + 1$ for u we get

$$H'(x) = \frac{dy}{dx} = 100\left(x^2 + x + 1\right)^{99}(2x+1)$$

Rule 7: The Chain Rule

If $h(x) = g[f(x)]$, then

$$h'(x) = \frac{d}{dx} g\left(f(x)\right) = g'\left(f(x)\right) f'(x)$$

Equivalently, if we write $y = h(x) = g(u)$, where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Chain Rule for Power Functions

Many composite functions have the special form

$$h(x) = g[f(x)]$$

where g is defined by the rule

$$g(x) = x^n \qquad (n\text{, a real number})$$

so that

$$h(x) = [f(x)]^n$$

In other words, the function h is given by the power of a function f.

Examples:

$$h(x) = \left(x^2 + x + 1\right)^{100} \qquad H(x) = \frac{1}{\left(5 - x^3\right)^3} \qquad G(x) = \sqrt{2x^2 + 3}$$

The General Power Rule

If the function *f* is differentiable and

$$h(x) = [f(x)]^n \qquad (n, \text{ a real number}),$$

then

$$h'(x) = \frac{d}{dx}[f(x)]^n = n[f(x)]^{n-1} f'(x)$$

Example 2

Find the derivative of $G(x) = \sqrt{x^2 + 1}$

Solution:

Rewrite as a power function: $G(x) = \left(x^2 + 1\right)^{1/2}$

Apply the general power rule:

$$G'(x) = \frac{1}{2}\left(x^2 + 1\right)^{-1/2} \frac{d}{dx}\left(x^2 + 1\right)$$

$$= \frac{1}{2}\left(x^2 + 1\right)^{-1/2}\left(2x\right)$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Example 3

Find the derivative of $f(x) = x^2(2x+3)^5$

Solution:

Apply the product rule and the general power rule:

$$f'(x) = x^2 \frac{d}{dx}(2x+3)^5 + (2x+3)^5 \frac{d}{dx}x^2$$

$$= x^2(5)(2x+3)^4(2) + (2x+3)^5(2)x$$

$$= 10x^2(2x+3)^4 + 2x(2x+3)^5$$

$$= 2x(2x+3)^4(5x+2x+3)$$

$$= 2x(2x+3)^4(7x+3)$$

Example 5

Find the derivative of $f(x) = \dfrac{1}{\left(4x^2 - 7\right)^2}$

Solution:

Rewrite as a power function: $f(x) = \left(4x^2 - 7\right)^{-2}$

Apply the general power rule:

$$f'(x) = -2\left(4x^2 - 7\right)^{-3}(8x)$$

$$= -\frac{16x}{\left(4x^2 - 7\right)^3}$$

Example 6

Find the derivative of $f(x)=\left(\dfrac{2x+1}{3x+2}\right)^3$

Solution:

Apply the general power rule and the quotient rule:

$$f'(x)=3\left(\frac{2x+1}{3x+2}\right)^2\frac{d}{dx}\left(\frac{2x+1}{3x+2}\right)$$

$$=3\left(\frac{2x+1}{3x+2}\right)^2\left[\frac{(3x+2)(2)-(2x+1)(3)}{(3x+2)^2}\right]$$

$$=3\left(\frac{2x+1}{3x+2}\right)^2\left[\frac{6x+4-6x-3}{(3x+2)^2}\right]=\frac{3(2x+1)^2}{(3x+2)^4}$$

Applied Problem 8 – *Arteriosclerosis*

Arteriosclerosis begins during childhood when plaque forms in the arterial walls, blocking the flow of blood through the arteries and leading to heart attacks, stroke and gangrene.

Applied Problem 8 – *Arteriosclerosis* cont'd

Suppose the idealized cross section of the aorta is circular with radius a cm and by year t the thickness of the plaque is

$$h = g(t) \text{ cm}$$

then the area of the opening is given by

$$A = \pi(a - h)^2 \text{ cm}^2$$

Further suppose the radius of an individual's artery is 1 cm ($a = 1$) and the thickness of the plaque in year t is given by

$$h = g(t) = 1 - 0.01(10{,}000 - t^2)^{1/2} \text{ cm}$$

Applied Problem 8 – *Arteriosclerosis* cont'd

Then we can use these functions for h and A

$h = g(t) = 1 - 0.01(10{,}000 - t^2)^{1/2}$ $A = f(h) = \pi(1 - h)^2$

to find a function that gives us the rate at which A is changing with respect to time by applying the chain rule:

$$\frac{dA}{dt} = \frac{dA}{dh}\cdot\frac{dh}{dt} = f'(h)\cdot g'(t)$$

$$= 2\pi(1-h)(-1)\left[-0.01\left(\frac{1}{2}\right)\left(10{,}000-t^2\right)^{-1/2}(-2t)\right]$$

$$= -2\pi(1-h)\left[\frac{0.01t}{\left(10{,}000-t^2\right)^{1/2}}\right]$$

$$= -\frac{0.02\pi(1-h)t}{\sqrt{10{,}000-t^2}}$$

Applied Problem 8 – *Arteriosclerosis* cont'd

For example, at age 50 (t = 50),

$$h = g(50) = 1 - 0.01(10,000 - 2500)^{1/2} \approx 0.134$$

So that

$$\frac{dA}{dt} = \frac{0.02\pi(1 - 0.134)50}{\sqrt{10,000 - 2500}} \approx -0.03$$

That is, the area of the arterial opening is decreasing at the rate of 0.03 cm^2 per year for a typical 50 year old.