3

# DIFFERENTIATION

# 3.6 Implicit Differentiation and Related Rates

# Differentiating Implicitly

Up to now we have dealt with functions in the form

$$y = f(x)$$

That is, the dependent variable $y$ has been expressed explicitly in terms of the independent variable $x$.

However, not all functions are expressed explicitly.

For example, consider

$$x^2y + y - x^2 + 1 = 0$$

This equation expresses $y$ implicitly as a function of $x$.

# Differentiating Implicitly

Solving for *y* in terms of *x* we get

$$(x^2 + 1)y = x^2 - 1$$

$$y = f(x) = \frac{x^2 - 1}{x^2 + 1}$$

which expresses *y* explicitly.

Now, consider the equation

$$y^4 - y^3 - y + 2x^3 - x = 8$$

# Differentiating Implicitly

With certain restrictions placed on *y* and *x*, this equation defines *y* as a function of *x*.

But in this case it is *difficult* to solve for *y* in order to express the function explicitly.

How do we compute *dy/dx* in this case?

The *chain rule* gives us a way to do this.

# Example 1

Consider the equation $y^2 = x$.

To find $dy/dx$, we differentiate both sides of the equation:

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(x)$$

Since $y$ is a function of $x$, we can rewrite $y = f(x)$ and find:

$$\begin{aligned}\frac{d}{dx}\left(y^2\right) &= \frac{d}{dx}[f(x)]^2 \\ &= 2f(x)f'(x) \\ &= 2y\frac{dy}{dx}\end{aligned}$$

Using chain rule

# Example 1

cont'd

Therefore the above equation is equivalent to:

$$2y\frac{dy}{dx}=1$$

Solving for *dy/dx* yields:

$$\frac{dy}{dx}=\frac{1}{2y}$$

# Steps for Differentiating Implicitly

To find *dy/dx* by implicit differentiation:

1. Differentiate both sides of the equation with respect to *x*.

   (Make sure that the derivative of any term involving *y* includes the factor *dy/dx*)

2. Solve the resulting equation for *dy/dx* in terms of *x* and *y*.

# Example 2

Find *dy/dx* for the equation $y^3 - y + 2x^3 - x = 8$

Solution:
Differentiating both sides and solving for *dy/dx* we get

$$\frac{d}{dx}\left(y^3 - y + 2x^3 - x\right) = \frac{d}{dx}(8)$$

$$\frac{d}{dx}\left(y^3\right) - \frac{d}{dx}(y) + \frac{d}{dx}\left(2x^3\right) - \frac{d}{dx}(x) = \frac{d}{dx}(8)$$

$$3y^2\frac{dy}{dx} - \frac{dy}{dx} + 6x^2 - 1 = 0$$

# Example 2 – *Solution*

cont'd

$$\frac{dy}{dx}\left(3y^2-1\right)=1-6x^2$$

$$\frac{dy}{dx}=\frac{1-6x^2}{3y^2-1}$$

# Example 4

Find *dy/dx* for the equation $x^2y^3 + 6x^2 = y + 12$

Then, find the value of *dy/dx* when $y = 2$ and $x = 1$.

Solution:

$$\frac{d}{dx}\left(x^2y^3\right) + \frac{d}{dx}\left(6x^2\right) = \frac{d}{dx}(y) + \frac{d}{dx}(12)$$

$$x^2 \cdot \frac{d}{dx}\left(y^3\right) + y^3 \cdot \frac{d}{dx}\left(x^2\right) + 12x = \frac{dy}{dx}$$

$$3x^2y^2\frac{dy}{dx} + 2xy^3 + 12x = \frac{dy}{dx}$$

# Example 4 – *Solution*

cont'd

$$\left(3x^2y^2 - 1\right)\frac{dy}{dx} = -2xy^3 - 12x$$

$$\frac{dy}{dx} = \frac{2xy^3 + 12x}{1 - 3x^2y^2}$$

Substituting $y = 2$ and $x = 1$ we find:

$$\frac{dy}{dx} = \frac{2xy^3 + 12x}{1 - 3x^2y^2}$$

$$= \frac{2(1)(2)^3 + 12(1)}{1 - 3(1)^2(2)^2}$$

$$= \frac{16 + 12}{1 - 12} = -\frac{28}{11}$$

# Example 5

Find *dy/dx* for the equation $\sqrt{x^2 + y^2} - x^2 = 5$

Solution:

$$\frac{d}{dx}\left(x^2 + y^2\right)^{1/2} - \frac{d}{dx}\left(x^2\right) = \frac{d}{dx}(5)$$

$$\frac{1}{2}\left(x^2 + y^2\right)^{-1/2}\left(2x + 2y\frac{dy}{dx}\right) - 2x = 0$$

$$\left(x^2 + y^2\right)^{-1/2}\left(2x + 2y\frac{dy}{dx}\right) = 4x$$

$$x + y\frac{dy}{dx} = 2x\left(x^2 + y^2\right)^{1/2}$$

# Example 5 – *Solution*

cont'd

$$y\frac{dy}{dx} = 2x\left(x^2 + y^2\right)^{1/2} - x$$

$$\frac{dy}{dx} = \frac{2x\left(x^2 + y^2\right)^{1/2} - x}{y}$$

# Related Rates

Implicit differentiation is a useful technique for solving a class of problems known as related-rate problems. Here are some guidelines to solve related-rate problems:

1. Assign a variable to each quantity.
2. Write the given values of the variables and their rate of change with respect to $t$.
3. Find an equation giving the relationship between the variables.
4. Differentiate both sides of the equation implicitly with respect to $t$.
5. Replace the variables and their derivatives by the numerical data found in step 2 and solve the equation for the required rate of change.

## Applied Example 6 – *Rate of Change of Housing Starts*

A study prepared for the National Association of Realtors estimates that the number of housing starts in the southwest, $N(t)$ (in millions), over the next 5 years is *related* to the mortgage rate $r(t)$ (percent per year) by the equation

$$9n^2 + r = 36$$

What is the rate of change of the number of housing starts with respect to time when the mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

# Applied Example 6 – *Solution*

We are given that $r = 11\%$ and $dr/dt = 1.5$ at a certain instant in time, and we are required to find $dN/dt$.

Substitute $r = 11$ into the given equation:

$$9N^2 + (11) = 36$$

$$N^2 = \frac{25}{9}$$

$$N = \frac{5}{3}$$

(rejecting the negative root)

# Applied Example 6 – *Solution*

cont'd

Differentiate the given equation implicitly on both sides with respect to *t*:

$$\frac{d}{dt}\left(9N^2\right)+\frac{d}{dt}\left(r\right)=\frac{d}{dt}\left(36\right)$$

$$18N\frac{dN}{dt}+\frac{dr}{dt}=0$$

# Applied Example 6 – *Solution*

cont'd

Substitute $N$ = 5/3 and $dr/dt$ = 1.5 into this equation and solve for $dN/dt$:

$$18\left(\frac{5}{3}\right)\frac{dN}{dt}+1.5=0$$

$$30\frac{dN}{dt}=-1.5$$

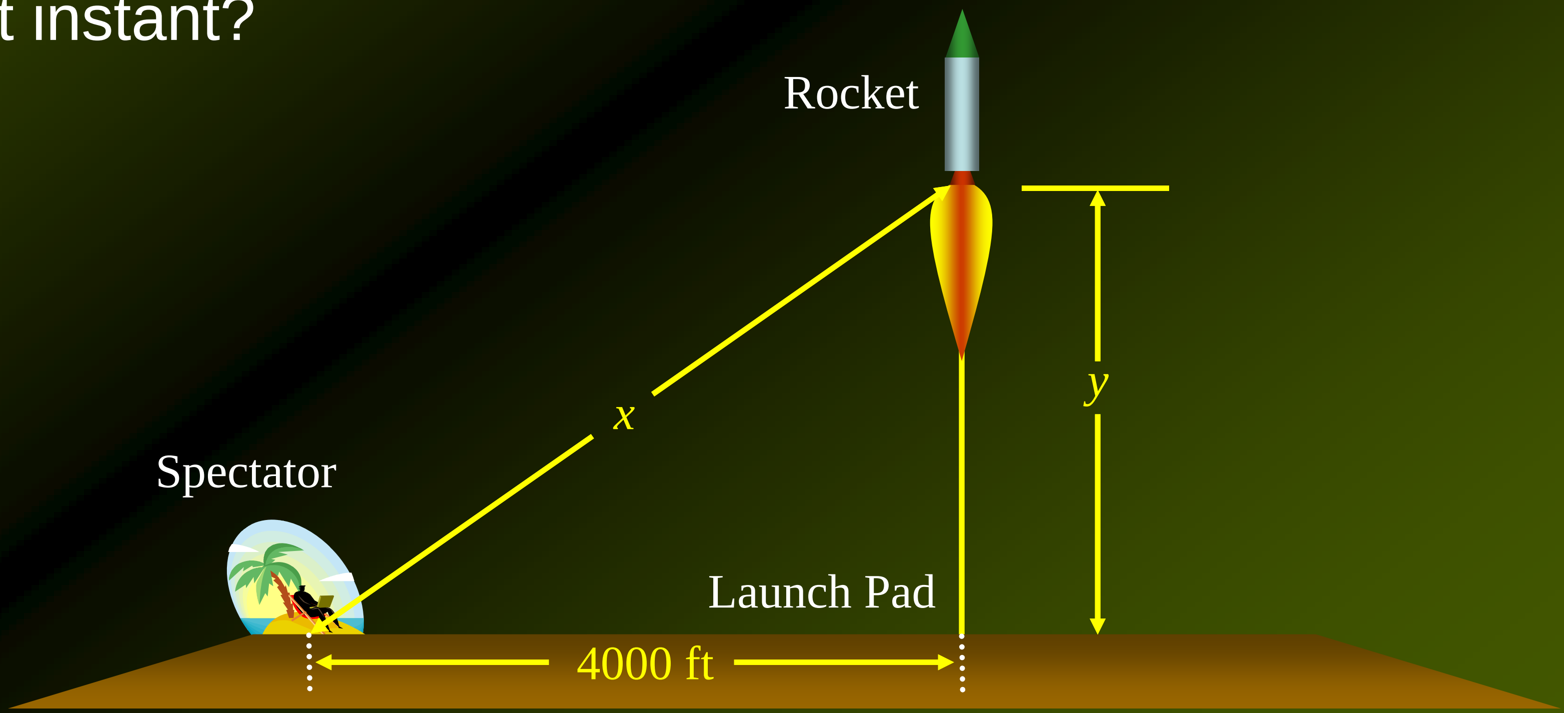
$$\frac{dN}{dt}=-\frac{1.5}{30}$$

$$\frac{dN}{dt}=-0.05$$

Thus, at the time under consideration, the number of housing starts is decreasing at rate of 50,000 units per year.

# Applied Example 8 – *Watching a Rocket Launch*

At a distance of 4000 feet from the launch site, a spectator is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 feet per second when it is at an altitude of 3000 feet, how fast is the distance between the rocket and the spectator changing at that instant?



# Applied Example 8 – *Solution*

1. Let

   $y$ = altitude of the rocket

   $x$ = distance between the rocket and the

   spectator at any time $t$.

2. We are told that at a certain instant in time

$$y = 3000 \quad \textbf{and} \quad \frac{dy}{dt} = 600$$

   and are asked to find $dx/dt$ at that instant.

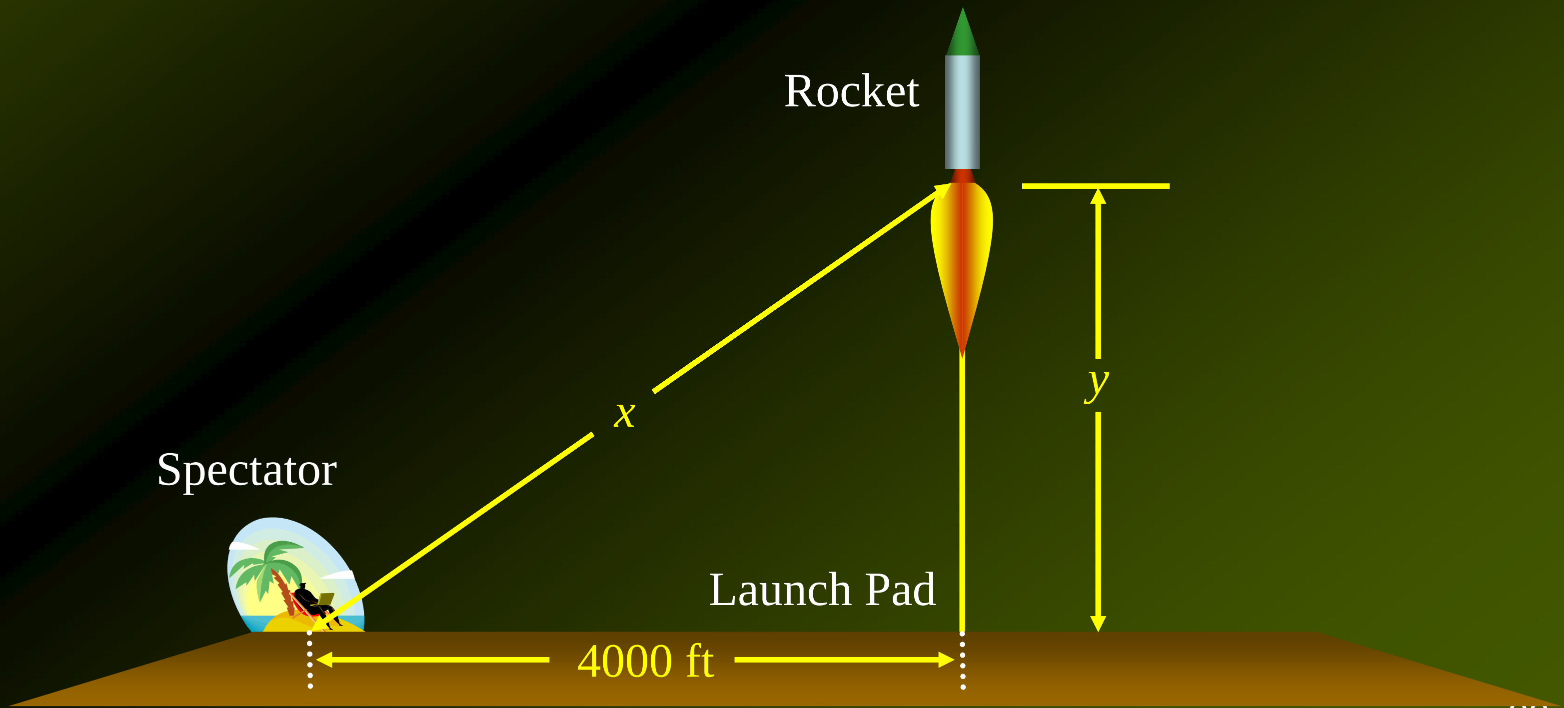
# Applied Example 8 – *Solution*

cont'd

3. Apply the Pythagorean theorem to the right triangle we find that

$$x^2 = y^2 + 4000^2$$

Therefore, when $y = 3000$, $x = \sqrt{3000^2 + 4000^2} = 5000$



# Applied Example 8 – *Solution*

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4. Differentiate $x^2 = y^2 + 4000^2$ with respect to $t$, obtaining

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

5. Substitute $x = 5000$, $y = 3000$, and $dy/dt = 600$, to find

$$2(5000)\frac{dx}{dt} = 2(3000)(600)$$

$$\frac{dx}{dt} = 360$$

Therefore, the distance between the rocket and the spectator is changing at a rate of 360 feet per second.