# 3 DIFFERENTIATION

# 3.7 Differentials

# Increments

Let $x$ denote a variable quantity and suppose $x$ changes from $x_1$ to $x_2$.

This change in $x$ is called the increment in $x$ and is denoted by the symbol $\Delta x$ (read "delta $x$").

Thus,

$$\Delta x = x_2 - x_1$$

# Example 1(a)

Find the increment in $x$ as $x$ changes from 3 to 3.2.

Solution:

Here, $x_1 = 3$ and $x_2 = 3.2$, so

$$Dx = x_2 - x_1$$

$$= 3.2 - 3 = 0.2$$

# Example 1(b)

Find the increment in $x$ as $x$ changes from 3 to 2.7.

Solution:

Here, $x_1 = 3$ and $x_2 = 2.7$, so

$$\Delta x = x_2 - x_1$$

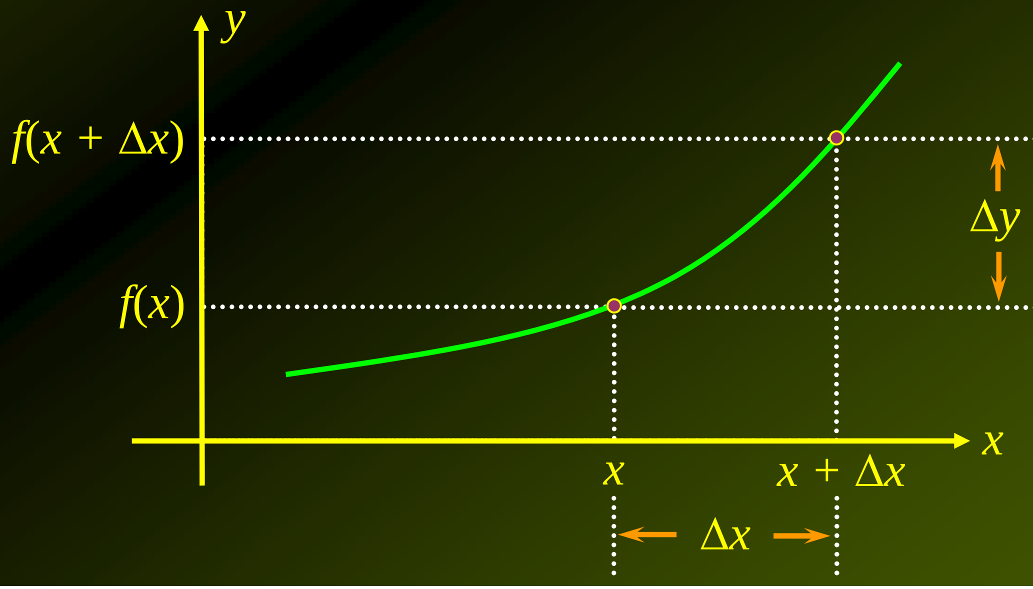
$$= 2.7 - 3 = -0.3$$

# Increments

Now, suppose two quantities, $x$ and $y$, are related by an equation $y = f(x)$, where $f$ is a function.

If $x$ changes from $x$ to $x + \Delta x$, then the corresponding change in $y$ is called the increment in $y$. It is denoted $\Delta y$ and is defined by

$$\Delta y = f(x + \Delta x) - f(x)$$



# Example 2

Let $y = x^3$. Find $\Delta x$ and $\Delta y$ when $x$ changes from

a. 2 to 2.01, and

b. from 2 to 1.98.

Solution:

a. Here, $\Delta x = 2.01 - 2 = 0.01$

Next, $\Delta y = f(x+\Delta x) - f(x) = f(2.01) - f(2)$

$$= (2.01)^3 - 2^3 = 8.120601 - 8 = 0.120601$$

b. Here, $\Delta x = 1.98 - 2 = -0.02$

Next, $\Delta y = f(x+\Delta x) - f(x) = f(1.98) - f(2)$

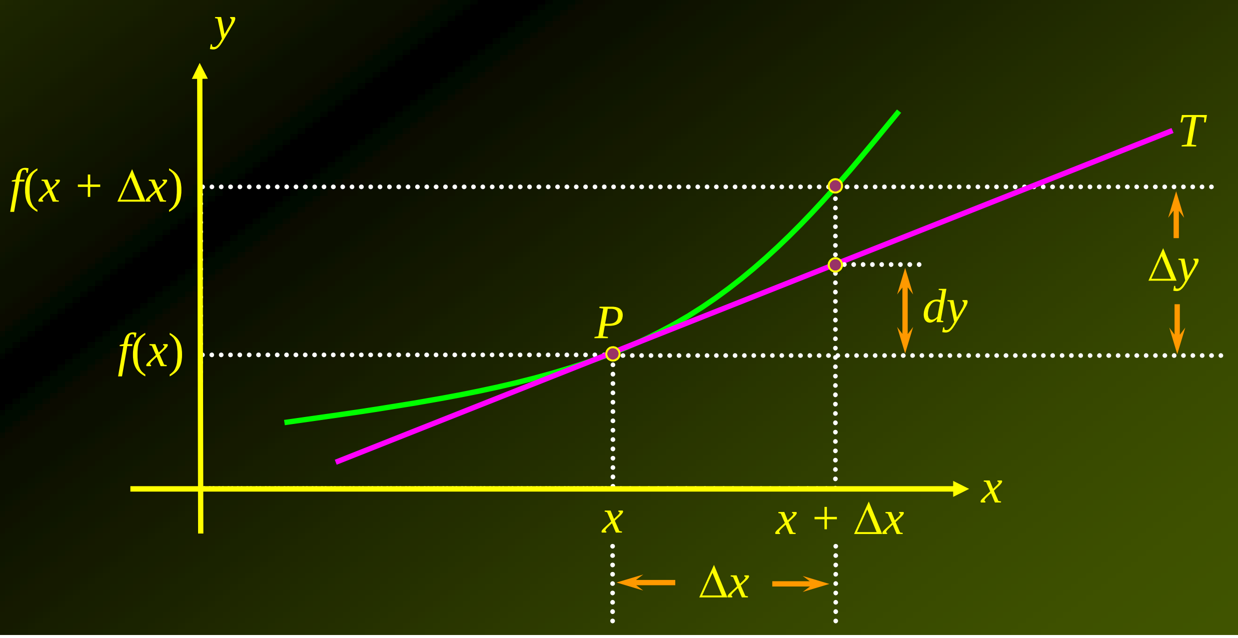
$$= (1.98)^3 - 2^3 = 7.762392 - 8 = -0.237608$$

# Differentials

We can obtain a relatively quick and simple way of approximating $\Delta y$, the change in $y$ due to small change $\Delta x$.

Observe below that near the point of tangency $P$, the tangent line $T$ is close to the graph of $f$.

Thus, if $\Delta x$ is small, then $dy$ is a good approximation of $\Delta y$.



# Differentials

Notice that the slope of *T* is given by $dy/\Delta x$ (rise over run).

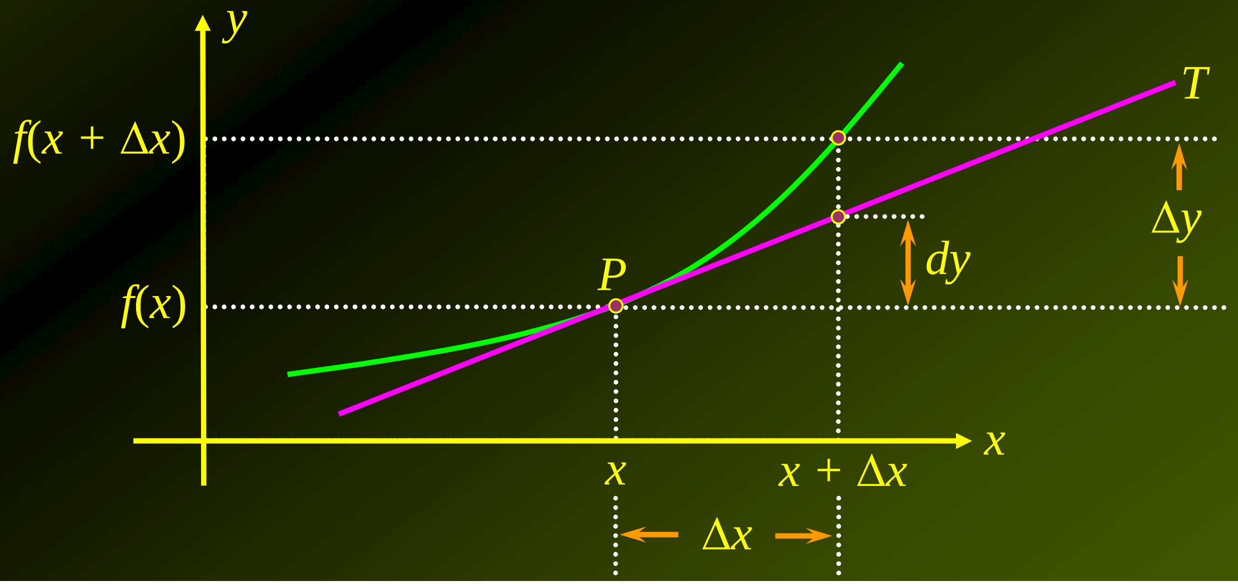
But the slope of *T* is given by $f'(x)$, so we have

$$dy/\Delta x = f'(x) \quad \text{or} \quad dy = f'(x)\,\Delta x$$

Thus, we have the approximation

$$\Delta y \approx dy = f'(x)\Delta x$$

The quantity *dy* is called the differential of *y*.



# The Differential

Let $y = f(x)$ define a differentiable function $x$. Then

1. The differential $dx$ of the independent variable $x$ is

$$dx = \Delta x$$

2. The differential $dy$ of the dependent variable $y$ is

$$dy = f'(x)\Delta x = f'(x)dx$$

# Example 4

Approximate the value of $\sqrt{26.5}$ using differentials.

Solution:

Let's consider the function $y = f(x) = \sqrt{x}$ .

Since 25 is the number nearest 26.5 whose square root is readily recognized, let's take $x = 25$.

We want to know the change in $y$, $\Delta y$, as $x$ changes from $x = 25$ to $x = 26.5$, an increase of $\Delta x = 1.5$.

So we find

$$\Delta y \approx dy = f'(x)\Delta x = \left( \frac{1}{2\sqrt{x}} \bigg|_{x=25} \right) \cdot (1.5) = \left( \frac{1}{10} \right)(1.5) = 0.15$$

# Example 4 – *Solution*

cont'd

Therefore,

$$\sqrt{26.5} - \sqrt{25} = \Delta y \approx 0.15$$

$$\sqrt{26.5} \approx \sqrt{25} + 0.15 = 5.15$$

## Applied Example 5 – *Effect of Speed on Vehicular Operating*

The total cost incurred in operating a certain type of truck on a 500-mile trip, traveling at an average speed of *v* mph, is estimated to be

$$C(v) = 125 + v + \frac{4500}{v}$$

dollars.

Find the approximate change in the total operating cost when the average speed is increased from 55 to 58 mph.

# Applied Example 5 – *Solution*

Total operating cost is given by

$$C(v) = 125 + v + \frac{4500}{v}$$

With $v$ = 55 an $\Delta v = dv$ = 3, we find

$$\begin{aligned} \Delta C \approx dC &= C'(v)dv \\ &= \left(1 - \frac{4500}{v^2}\right)\Bigg|_{v=55} (3) \\ &= \left(1 - \frac{4500}{3025}\right)(3) \approx -1.46 \end{aligned}$$

so the total operating cost is found to decrease by $1.46.

This might explain why so many independent truckers often exceed the 55 mph speed limit.