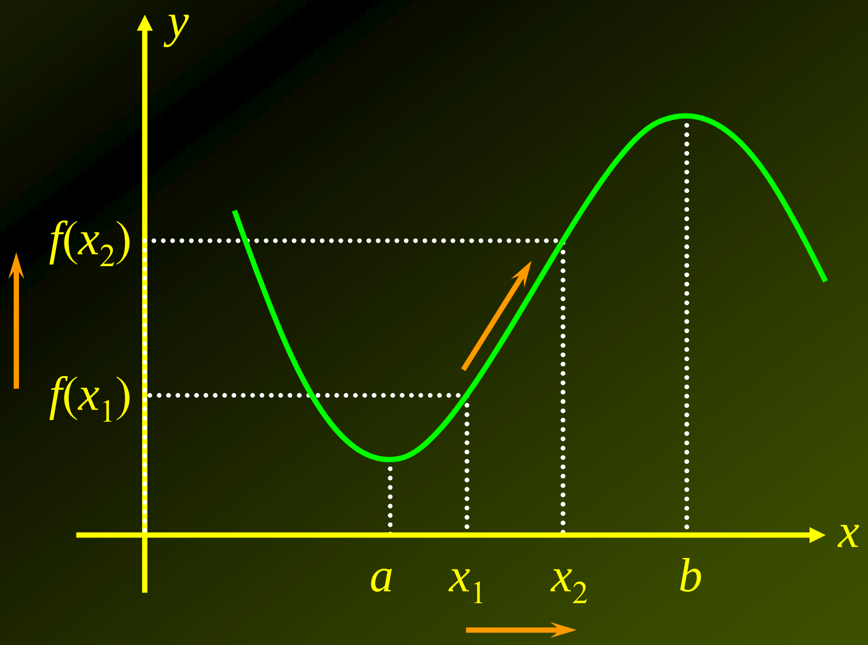
4 APPLICATIONS OF THE DERIVATIVE

4.1 Applications of the First Derivative

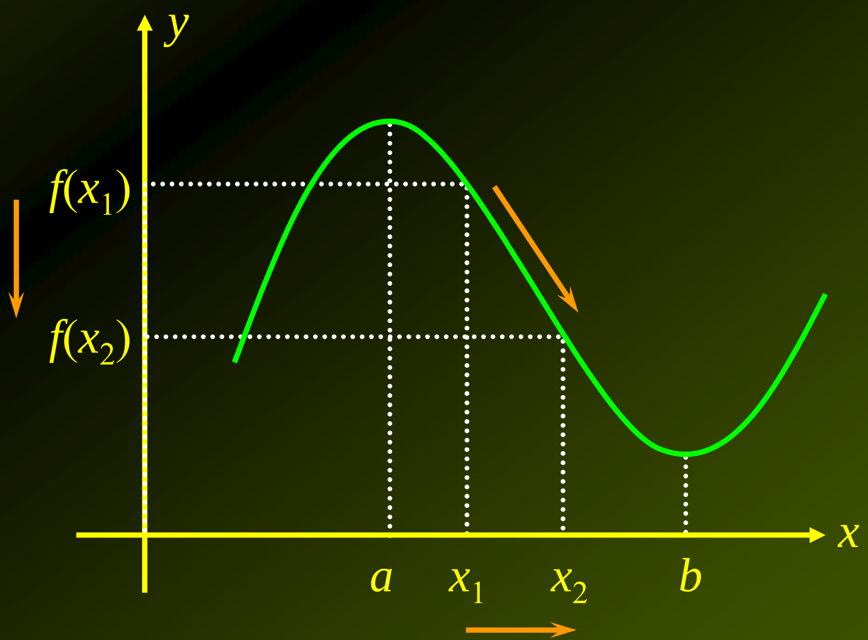
Increasing and Decreasing Functions

A function f is increasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),
$f(x_1) < f(x_2)$ wherever $x_1 < x_2$.



Increasing and Decreasing Functions

A function f is decreasing on an interval (a, b) if for any two numbers x_1 and x_2 in (a, b),
$f(x_1) > f(x_2)$ wherever $x_1 < x_2$.



Theorem 1

If $f'(x) > 0$ for each value of x in an interval (a, b), then f is increasing on (a, b).

If $f'(x) < 0$ for each value of x in an interval (a, b), then f is decreasing on (a, b).

If $f'(x) = 0$ for each value of x in an interval (a, b), then f is constant on (a, b).

Example 1

Find the interval where the function $f(x) = x^2$ is increasing and the interval where it is decreasing.

Solution:

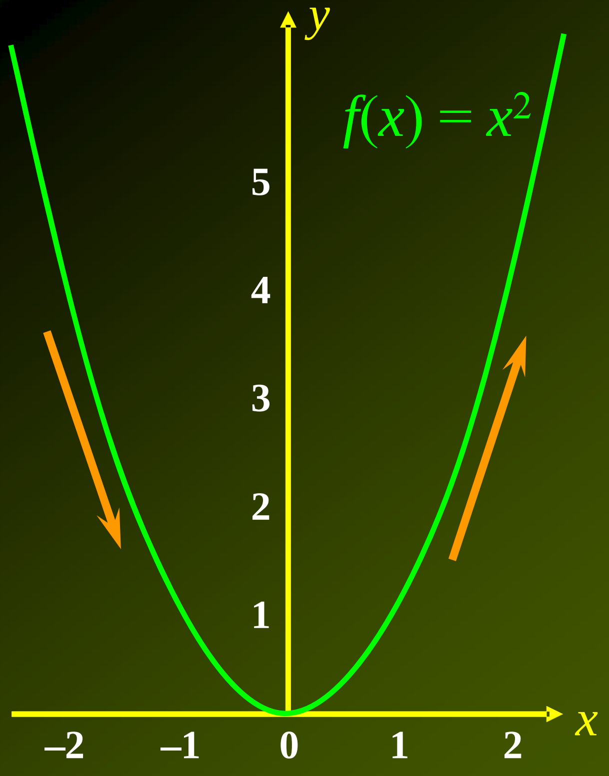
The derivative of $f(x) = x^2$

is $f'(x) = 2x$.

$f'(x) = 2x > 0$ if $x > 0$

and $f'(x) = 2x < 0$ if $x < 0$.

Thus, f is increasing on the interval $(0, \infty)$ and decreasing on the interval $(-\infty, 0)$.



Determining the Intervals Where a Function is Increasing or Decreasing

1. Find all the values of x for which $f'(x) = 0$ or f' is discontinuous and identify the open intervals determined by these numbers.

2. Select a test number c in each interval found in step 1 and determine the sign of $f'(c)$ in that interval.

   a. If $f'(c) > 0$, f is increasing on that interval.

   b. If $f'(c) < 0$, f is decreasing on that interval.

Example 2

Determine the intervals where the function

$$f(x) = x^3 - 3x^2 - 24x + 32$$

is increasing and where it is decreasing.

Solution:

1. Find f' and solve for $f'(x) = 0$:

$$f'(x) = 3x^2 - 6x - 24 = 3(x + 2)(x - 4) = 0$$

Thus, the zeros of f' are $x = -2$ and $x = 4$.

These numbers divide the real line into the intervals $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$.

Example 2 – *Solution*

cont'd

2. To determine the sign of $f'(x)$ in the intervals we found $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$, we compute $f'(c)$ at a convenient test point in each interval.

   Lets consider the values -3, 0, and 5:

   $$f'(-3) = 3(-3)^2 - 6(-3) - 24 = 27 + 18 - 24 = 21 > 0$$

   $$f'(0) = 3(0)^2 - 6(0) - 24 = 0 + 0 - 24 = -24 < 0$$

   $$f'(5) = 3(5)^2 - 6(5) - 24 = 75 - 30 - 24 = 21 > 0$$

   Thus, we conclude that f is increasing on the intervals $(-\infty, -2)$, $(4, \infty)$, and is decreasing on the interval $(-2, 4)$.

Example 2 – *Solution*

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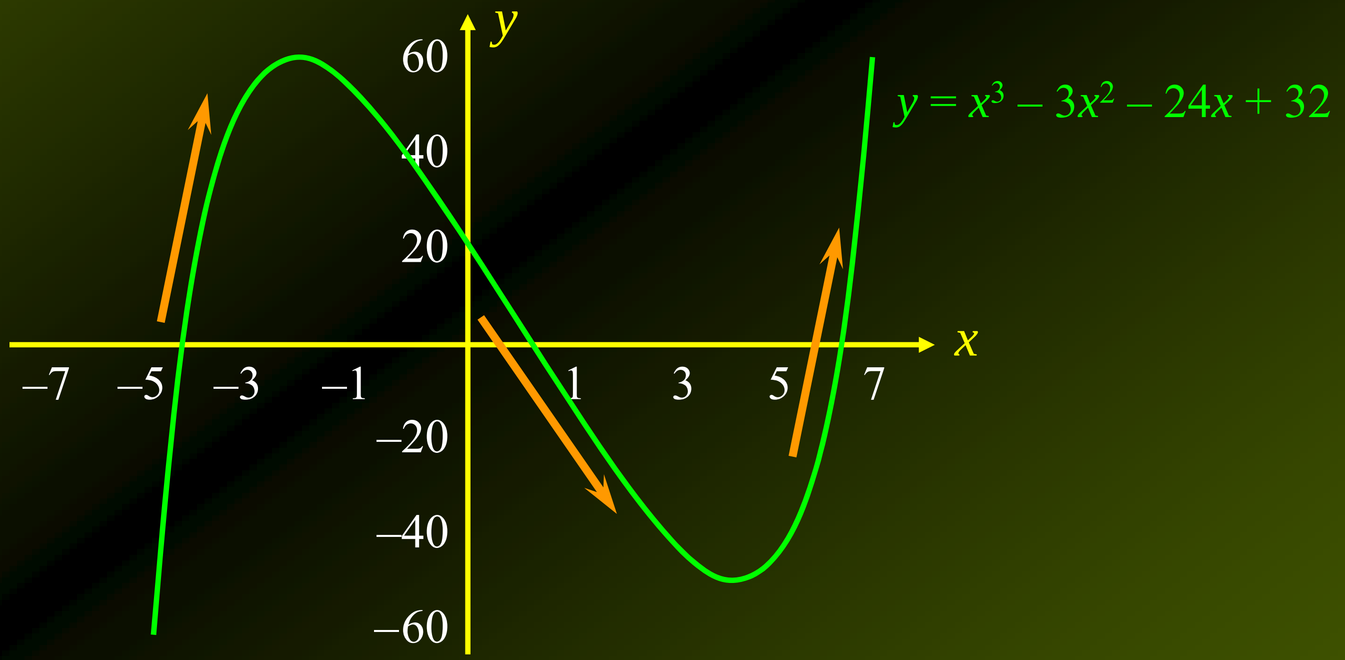
So, *f* increases on $(-\infty, -2)$, $(4, \infty)$, and decreases on $(-2, 4)$:



Example 4

Determine the intervals where $f(x) = x + \frac{1}{x}$ is increasing and where it is decreasing.

Solution:

1. Find f' and solve for $f'(x) = 0$:

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2} = 0$$

$f'(x) = 0$ when the numerator is equal to zero, so:

$$x^2 - 1 = 0$$
$$x^2 = 1$$
$$x = \pm 1$$

Thus, the zeros of f' are $x = -1$ and $x = 1$.

Example 4 – *Solution*

cont'd

Also note that f' is not defined at $x = 0$, so we have four intervals to consider: $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$.

2. To determine the sign of $f'(x)$ in the intervals we found $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, and $(1, \infty)$, we compute $f'(c)$ at a convenient test point in each interval.

Lets consider the values -2, $-1/2$, $1/2$, and 2:

$$f'(-2) = 1 - \frac{1}{(-2)^2} = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

So f is increasing in the interval $(-\infty, -1)$.

Example 4 – *Solution*

cont'd

$$f'(-\tfrac{1}{2}) = 1 - \frac{1}{\left(-\tfrac{1}{2}\right)^2} = 1 - \frac{1}{\tfrac{1}{4}} = 1 - 4 = -3 < 0$$

So *f* is decreasing in the interval (– 1, 0).

$$f'(\tfrac{1}{2}) = 1 - \frac{1}{\left(\tfrac{1}{2}\right)^2} = 1 - \frac{1}{\tfrac{1}{4}} = 1 - 4 = -3 < 0$$

So *f* is decreasing in the interval (0, 1).

$$f'(2) = 1 - \frac{1}{\left(2\right)^2} = 1 - \frac{1}{4} = \frac{3}{4} > 0$$

So *f* is increasing in the interval (1, ∞).

Example 4 – *Solution*

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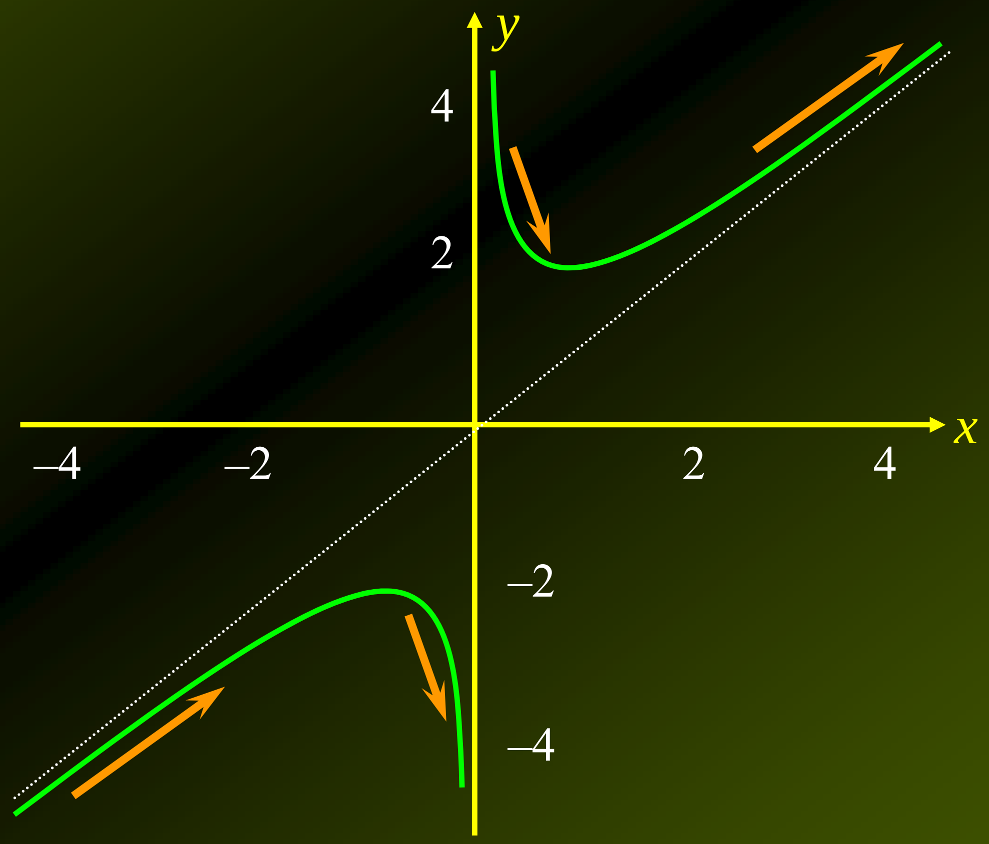
Thus, f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and decreasing on $(-1, 0)$ and $(0, 1)$:

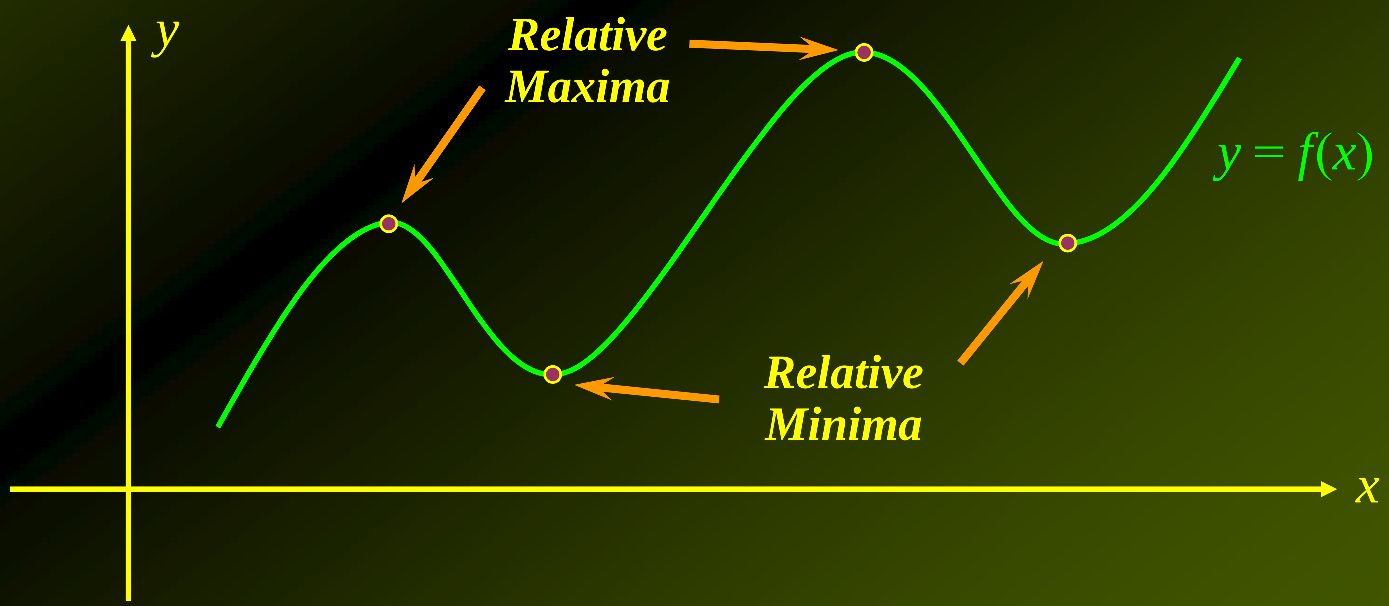


Relative Extrema

The first derivative may be used to help us *locate* high points and low points on the graph of *f*:

- High points are called relative maxima
- Low points are called relative minima.

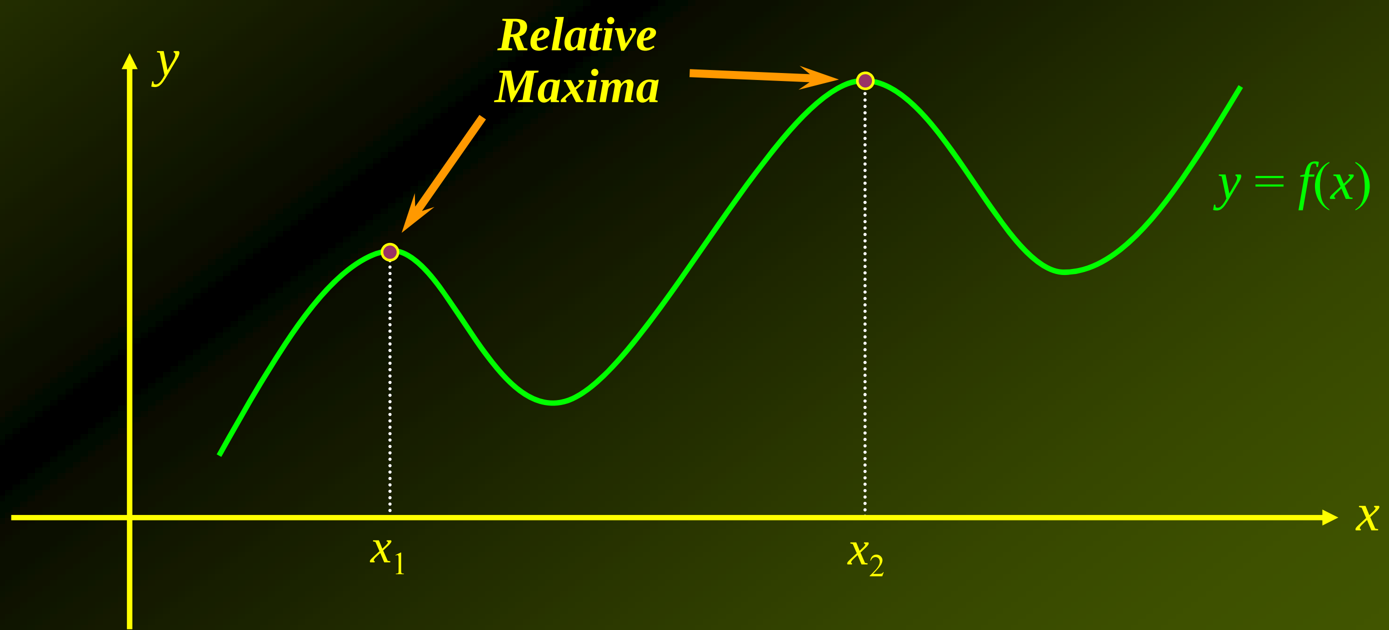
Both high and low points are called relative extrema.



Relative Extrema

Relative Maximum

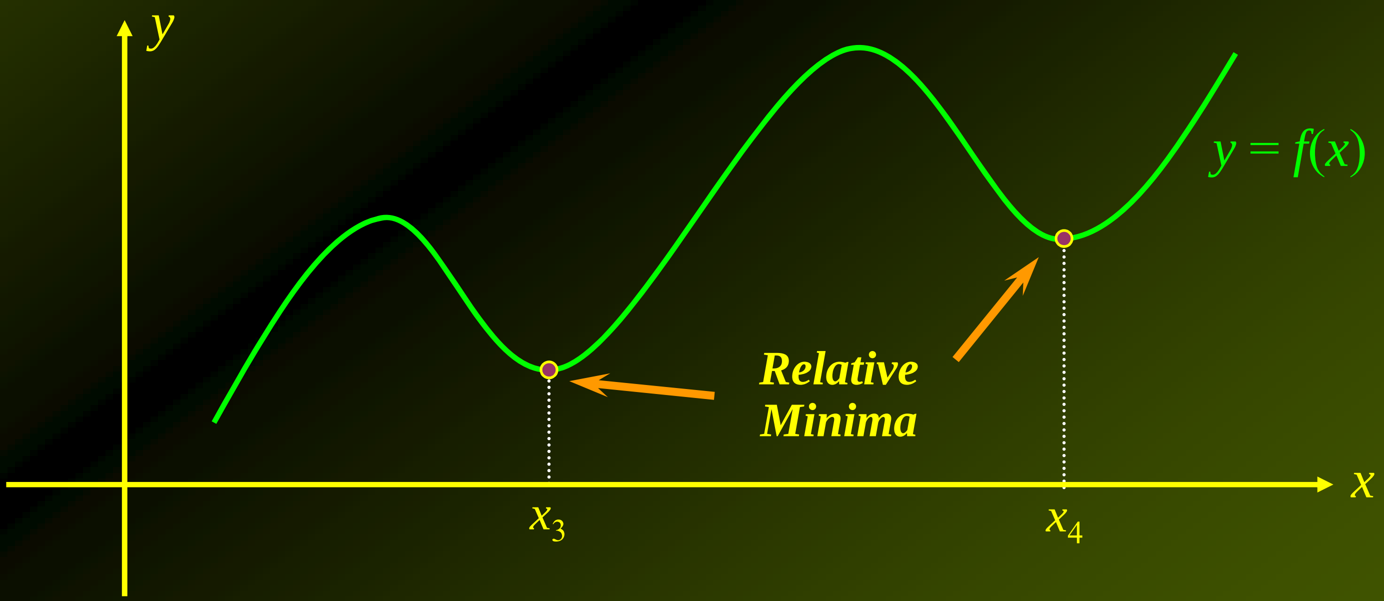
A function f has a relative maximum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \leq f(c)$ for all x in (a, b).



Relative Extrema

Relative Minimum

A function f has a relative minimum at $x = c$ if there exists an open interval (a, b) containing c such that $f(x) \geq f(c)$ for all x in (a, b).

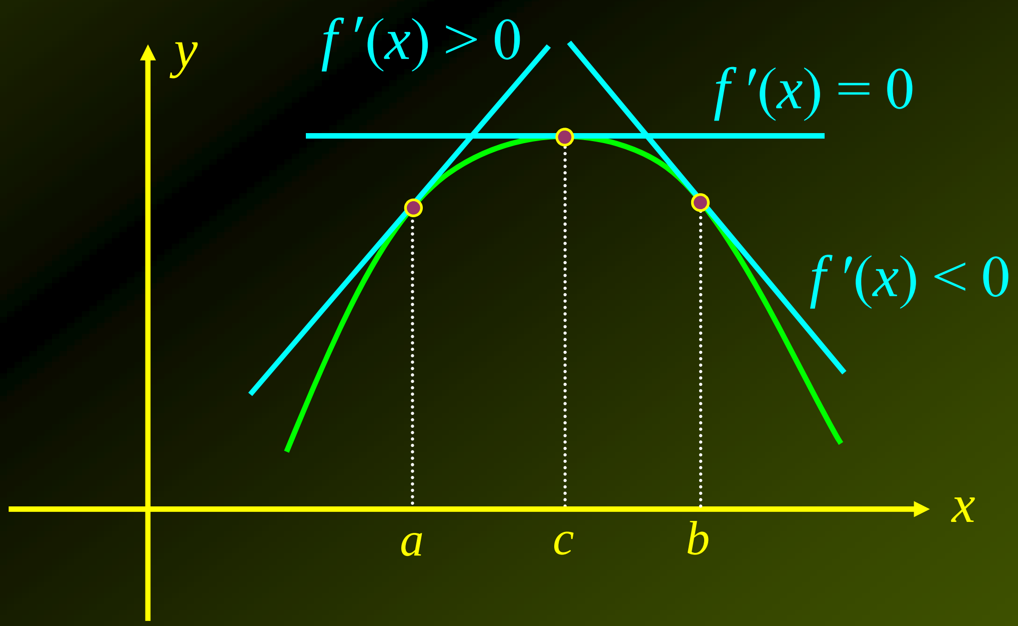


Finding Relative Extrema

Suppose that f has a relative maximum at c.

The slope of the tangent line to the graph must change from positive to negative as x increases.

Therefore, the tangent line to the graph of f at point $(c, f(c))$ must be horizontal, so that $f'(x) = 0$ or $f'(x)$ is undefined.

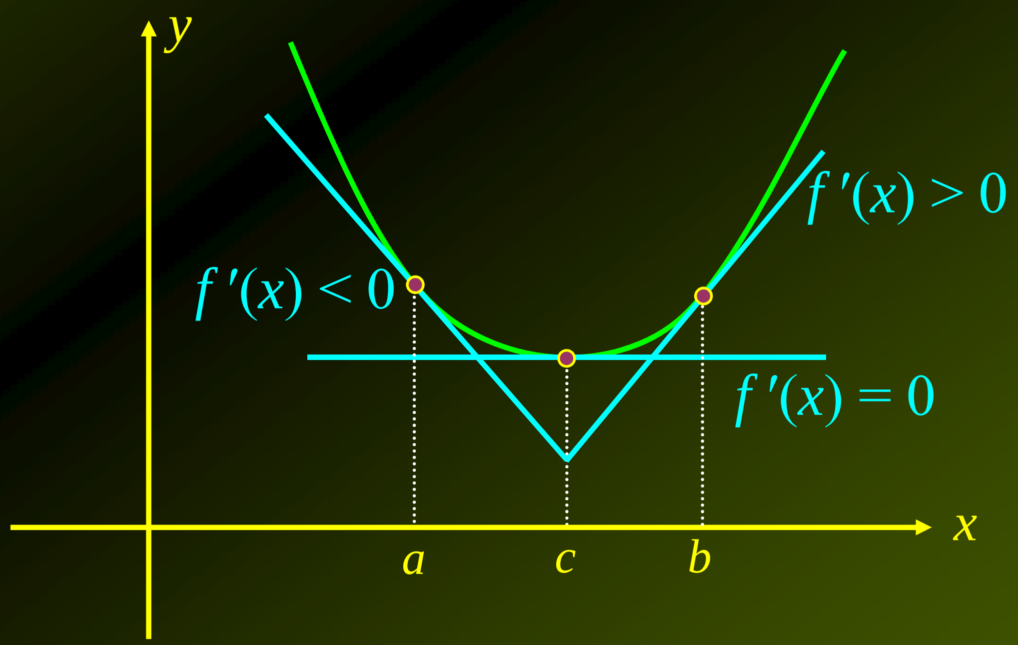


Finding Relative Extrema

Suppose that *f* has a relative minimum at *c*.

The slope of the tangent line to the graph must change from negative to positive as *x* increases.

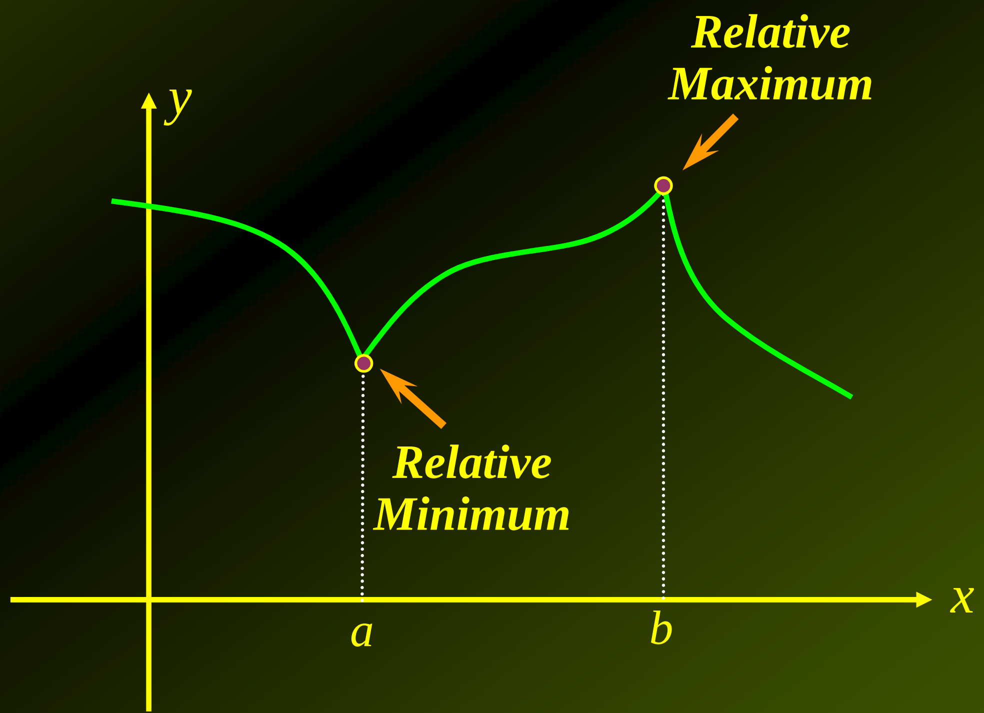
Therefore, the tangent line to the graph of *f* at point $(c, f(c))$ must be horizontal, so that $f'(x) = 0$ or $f'(x)$ is undefined.



Finding Relative Extrema

In some cases a derivative does not exist for particular values of *x*.

Extrema may exist at such points, as the graph below shows:



Critical Numbers

We refer to a number in the domain of *f* that *may* give rise to a relative extremum as a critical number.

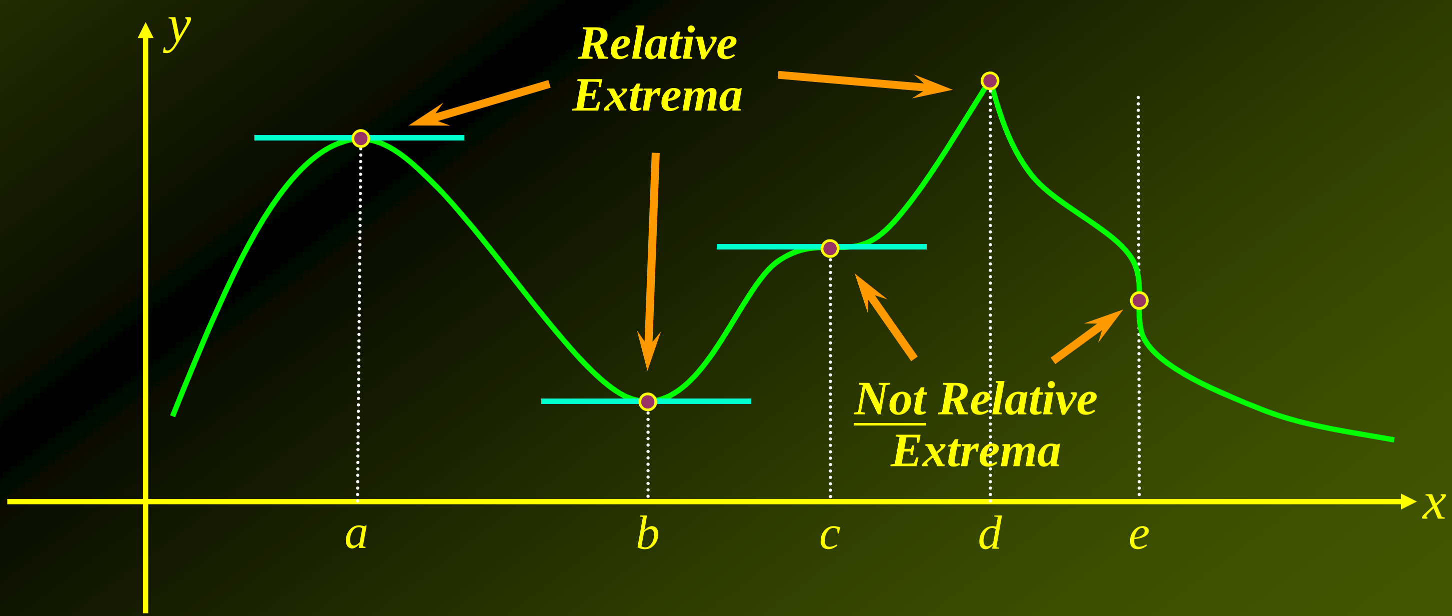
Critical number of *f*

A critical number of a function *f* is any number x in the domain of *f* such that $f'(x) = 0$ or $f'(x)$ does not exist.

Critical Numbers

The graph below shows us several critical numbers.
At points *a*, *b*, and *c*, $f'(x) = 0$.

There is a corner at point *d*, so $f'(x)$ does not exist there. The tangent to the curve at point *e* is vertical, so $f'(x)$ does not exist there either. Note that points *a*, *b*, and *d* are relative extrema, while points *c* and *e* are not.



The First Derivative Test

Procedure for Finding Relative Extrema of a Continuous Function f

1. Determine the critical numbers of f.
2. Determine the sign of $f'(x)$ to the left and right of each critical point.
   a. If $f'(x)$ changes sign from positive to negative as we move across a critical number c, then $f(c)$ is a relative maximum.
   b. If $f'(x)$ changes sign from negative to positive as we move across a critical number c, then $f(c)$ is a relative minimum.
   c. If $f'(x)$ does not change sign as we move across a critical number c, then $f(c)$ is not a relative extremum.

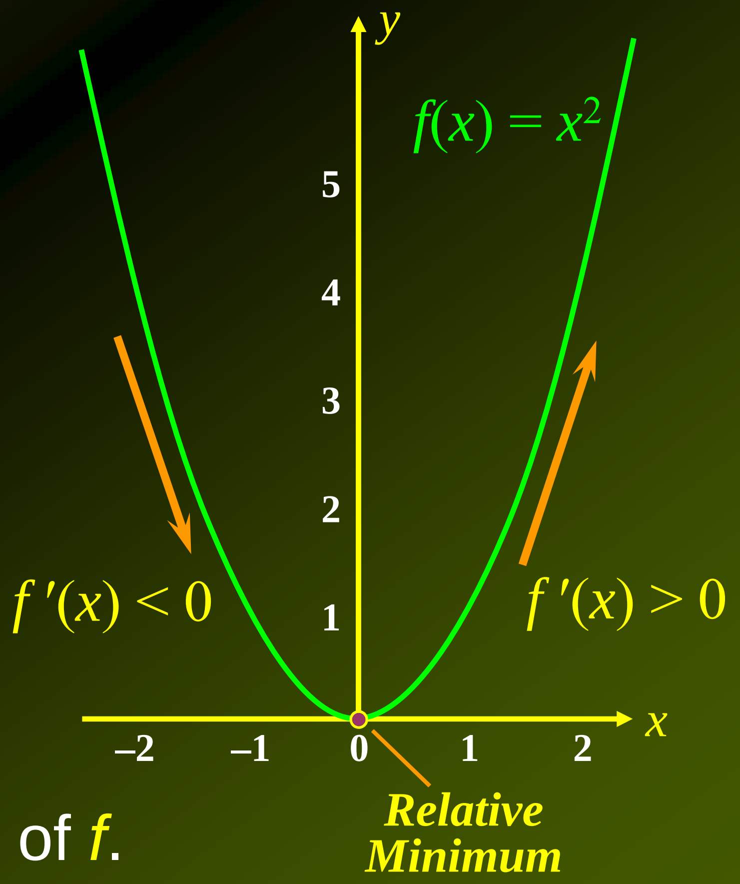
Example 5

Find the relative maxima and minima of $f(x) = x^2$

Solution:

The derivative of f is $f'(x) = 2x$.

Setting $f'(x) = 0$ yields $x = 0$ as the only critical number of f.
Since $f'(x) < 0$ if $x < 0$
and $f'(x) > 0$ if $x > 0$
we see that $f'(x)$ changes sign from negative to positive as we move across 0.



Thus, $f(0) = 0$ is a relative minimum of f.

Example 6

Find the relative maxima and minima of $f(x) = x^{2/3}$

Solution:

The derivative of f is $f'(x) = 2/3x^{-1/3}$.

$f'(x)$ is not defined at $x = 0$, is continuous everywhere else, and is never equal to zero in its domain.

Thus $x = 0$ is the only critical number of f.

Example 6 – *Solution*

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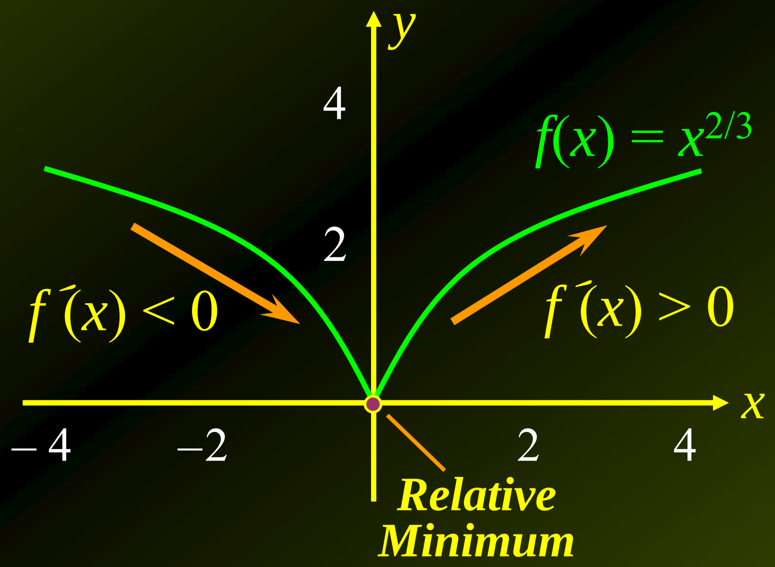
Since $f'(x) < 0$ if $x < 0$ and $f'(x) > 0$ if $x > 0$ we see that $f'(x)$ changes sign from negative to positive as we move across 0.



Thus, $f(0) = 0$ is a relative minimum of f.

Example 7

Find the relative maxima and minima of

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Solution:

The derivative of *f* and equate to zero:

$$f'(x) = 3x^2 - 6x - 24 = 0$$

$$3(x^2 - 2x - 8) = 0$$

$$3(x - 4)(x + 2) = 0$$

The zeros of $f'(x)$ are $x = -2$ and $x = 4$.

$f'(x)$ is defined everywhere, so $x = -2$ and $x = 4$ are the only critical numbers of *f*.

Example 7 – *Solution*

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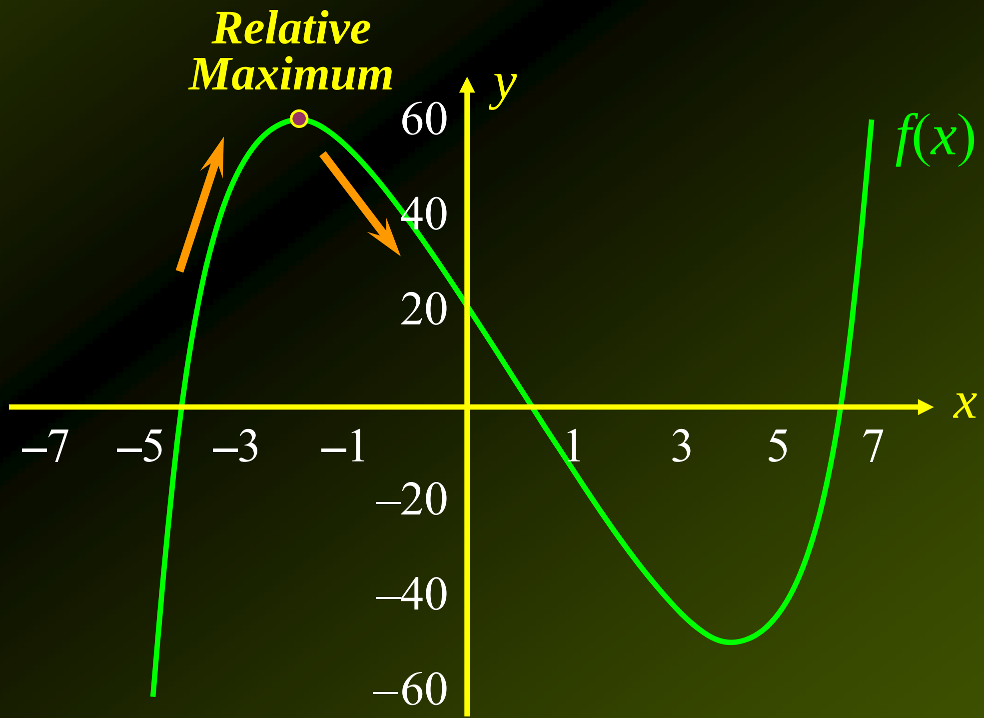
Since $f'(x) > 0$ if $x < -2$ and $f'(x) < 0$ if $0 < x < 4$, we see that $f'(x)$ changes sign from positive to negative as we move across -2.

Thus, $f(-2) = 60$ is a relative maximum.



Example 7 – *Solution*

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Since $f'(x) < 0$ if $0 < x < 4$ and $f'(x) > 0$ if, $x > 4$ we see that $f'(x)$ changes sign from negative to positive as we move across 4.

Thus, $f(4) = -48$ is a relative minimum.