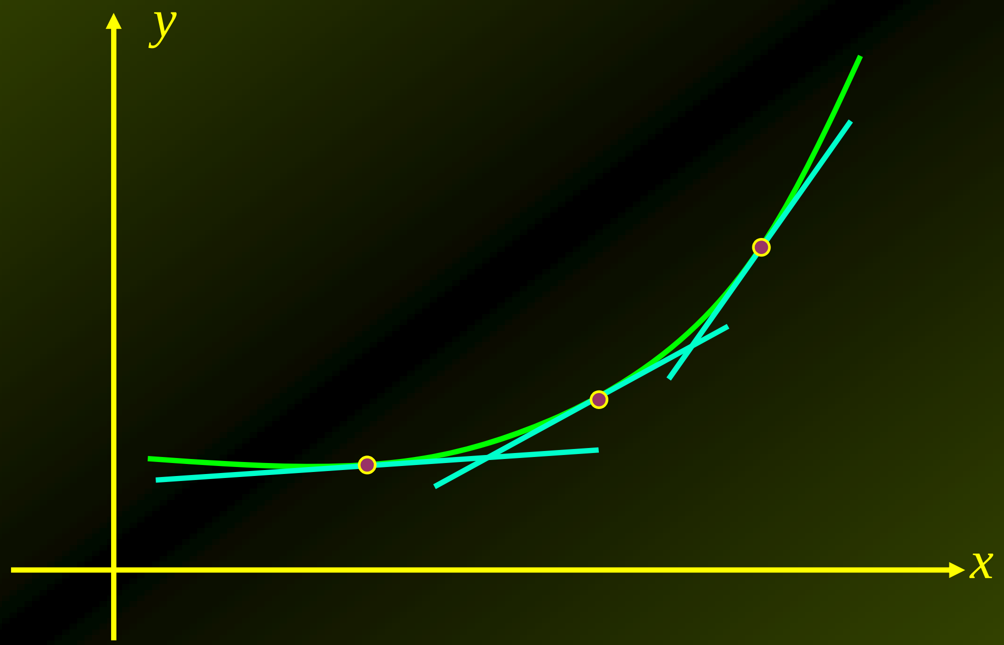
4

APPLICATIONS OF THE DERIVATIVE

4.2 Applications of the Second Derivative

Concavity

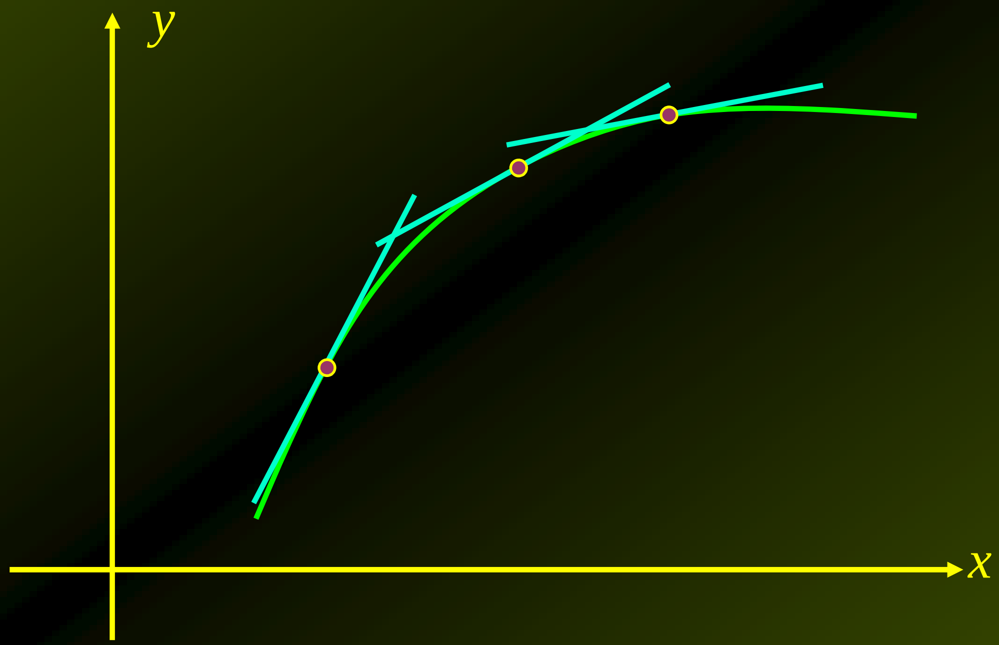
A curve is said to be concave upwards when the slope of tangent line to the curve is increasing:



Thus, if f is differentiable on an interval (a, b), then f is concave upwards on (a, b) if f' is increasing on (a, b).

Concavity

A curve is said to be concave downwards when the slope of tangent line to the curve is decreasing:



Thus, if f is differentiable on an interval (a, b), then f is concave downwards on (a, b) if f' is decreasing on (a, b).

Theorem 2

Recall that $f''(x)$ measures the rate of change of the slope $f'(x)$ of the tangent line to the graph of f at the point $(x, f(x))$.

Thus, we can use $f''(x)$ to determine the concavity of f.

a. If $f''(x) > 0$ for each value of x in (a, b), then f is concave upward on (a, b).

b. If $f''(x) < 0$ for each value of x in (a, b), then f is concave downward on (a, b).

Steps in Determining the Concavity of f

1. Determine the values of x for which f'' is zero or where f'' is not defined, and identify the open intervals determined by these numbers.
2. Determine the sign of f'' in each interval found in step 1. To do this compute $f''(c)$, where c is any conveniently chosen test number in the interval.
   a. If $f''(c) > 0$, f is concave upward on that interval.
   b. If $f''(c) < 0$, f is concave downward on that interval.

Example 1

Determine where the function $f(x) = x^3 - 3x^2 - 24x + 32$ is concave upward and where it is concave downward.

Solution:

Here, $f'(x) = 3x^2 - 6x - 24$ and $f''(x) = 6x - 6$

Setting $f''(c) = 0$ we find

$$f''(x) = 6x - 6 = 0$$

$$6(x - 1) = 0$$

which gives $x = 1$.

Example 1 – *Solution*

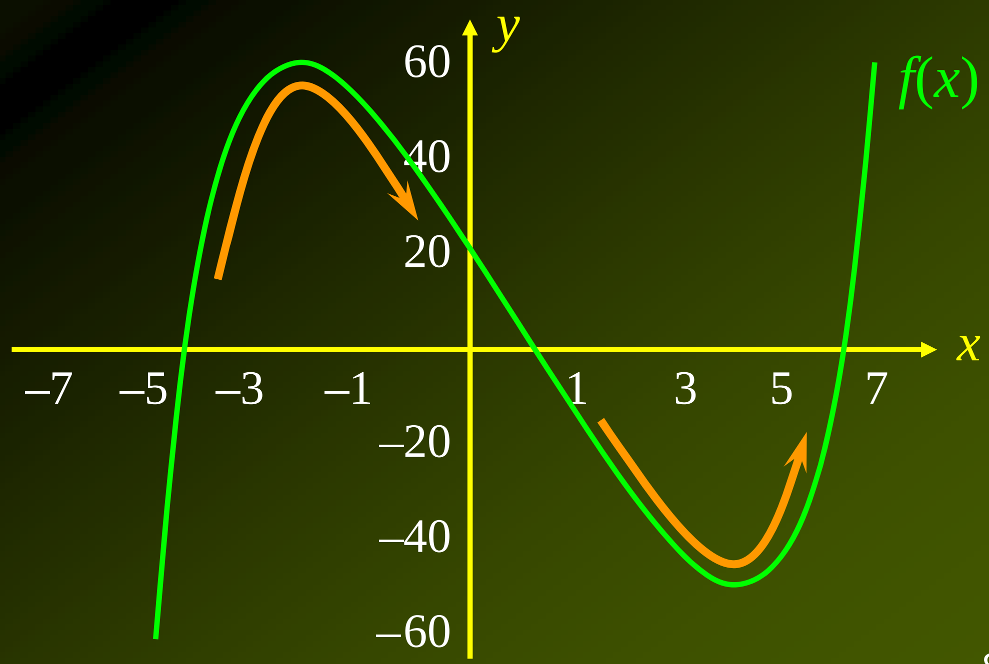
cont'd

So we consider the intervals $(-\infty, 1)$ and $(1, \infty)$:

1. $f''(x) < 0$ when $x < 1$, so f is concave downward on $(-\infty, 1)$.

2. $f''(x) > 0$ when $x > 1$, so f is concave upward on $(1, \infty)$.

The graph confirms that f is concave downward on $(-\infty, 1)$ and concave upward on $(1, \infty)$:



Example 2

Determine the intervals where the function $f(x) = x + \frac{1}{x}$ is concave upward and concave downward.

Solution:

Here, $f'(x) = 1 - \frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3}$

So, f'' cannot be zero and is not defined at $x = 0$.

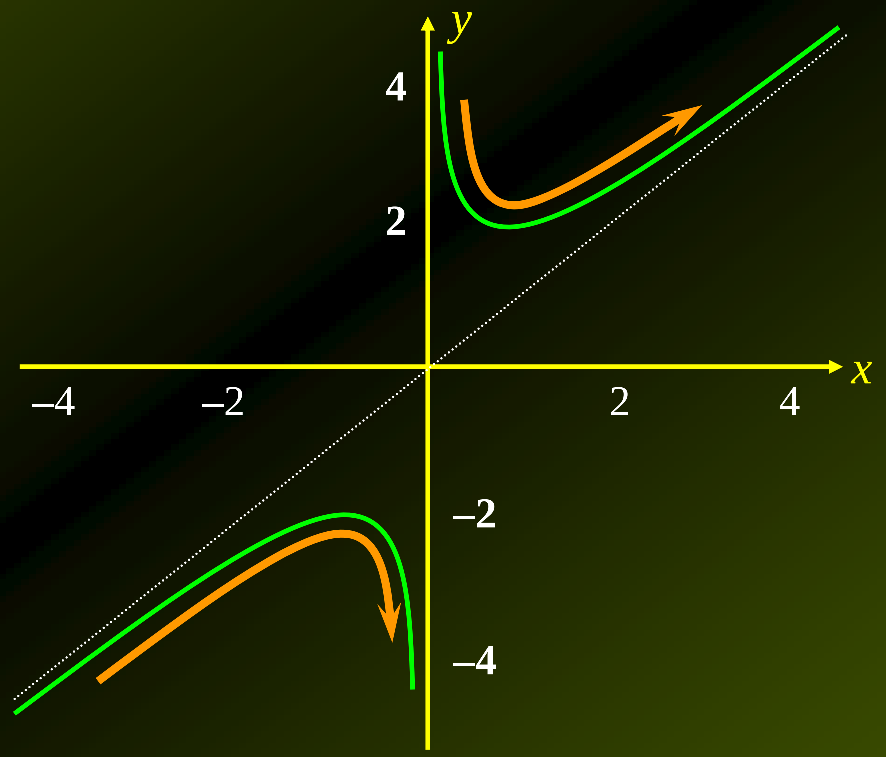
So we consider the intervals $(-\infty, 0)$ and $(0, \infty)$:

1. $f''(x) < 0$ when $x < 0$, so f is concave downward on $(-\infty, 0)$.
2. $f''(x) > 0$ when $x > 0$, so f is concave upward on $(0, \infty)$.

Example 2 – *Solution*

cont'd

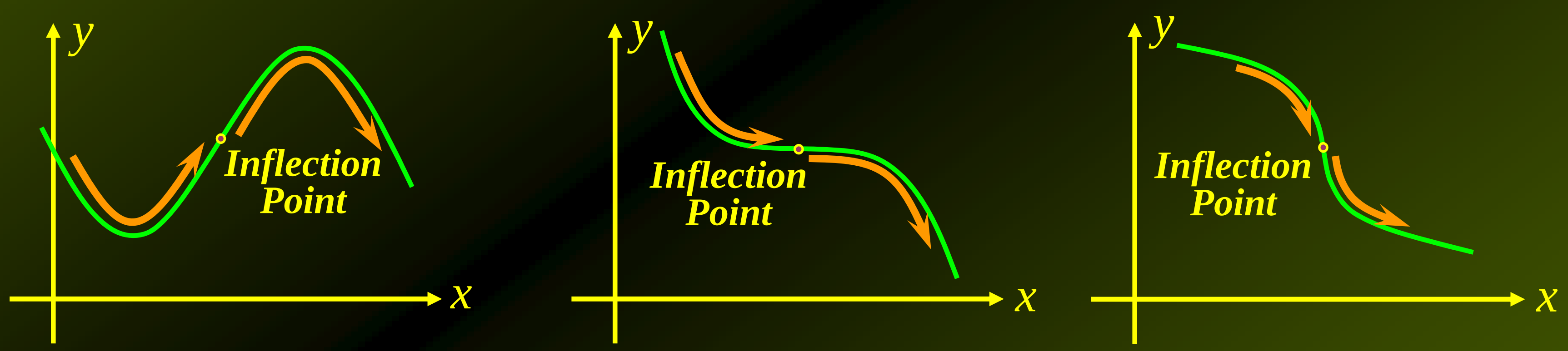
The graph confirms that *f* is concave downward on $(-\infty, 0)$ and concave upward on $(0, \infty)$:



Inflection Point

A point on the graph of a continuous function where the tangent line exists and where the concavity changes is called an inflection point.

Examples:



Finding Inflection Points

To find inflection points:

1. Compute $f''(x)$.
2. Determine the numbers in the domain of f for which $f''(x) = 0$ or $f''(x)$ does not exist.
3. Determine the sign of $f''(x)$ to the left and right of each number c found in step 2.

   If there is a change in the sign of $f''(x)$ as we move across $x = c$, then $(c, f(c))$ is an inflection point of f.

Example 3

Find the points of inflection of the function $f(x) = x^3$

Solution:

We have $f'(x) = 3x^2$ and $f''(x) = 6x$

So, f'' is continuous everywhere and is zero for $x = 0$.

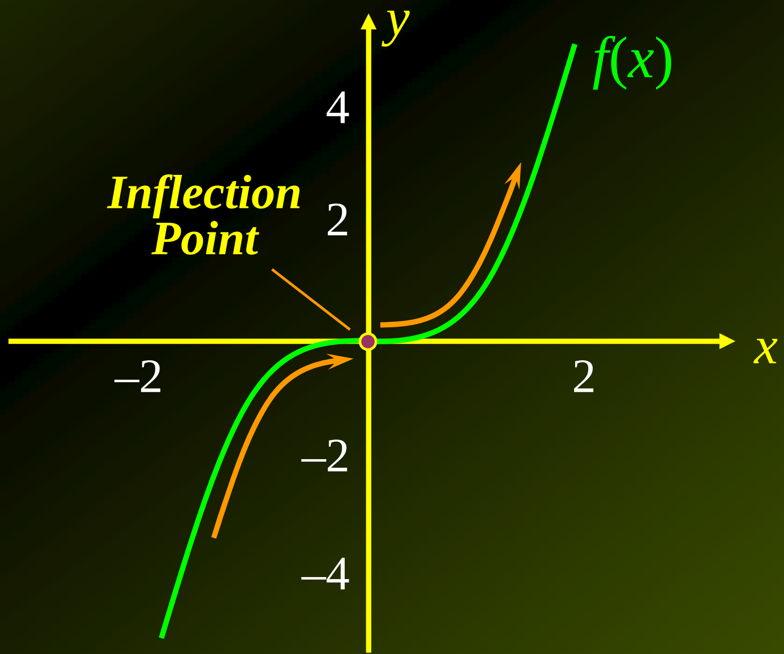
We see that $f''(x) < 0$ when $x < 0$, and $f''(x) > 0$ when $x > 0$.

Example 3 – *Solution*

cont'd

Thus, we find that the graph of f :

- Has one and only inflection point at $f(0) = 0$.
- Is concave downward on the interval $(-\infty, 0)$.
- Is concave upward on the interval $(0, \infty)$.



Example 4

Find the points of inflection of the function $f(x) = (x-1)^{5/3}$

Solution:

We have $f'(x) = \frac{5}{3}(x-1)^{2/3}$ and $f''(x) = \frac{10}{9(x-1)^{1/3}}$

So, f'' is not defined at $x = 1$, and f'' is never equal to zero.

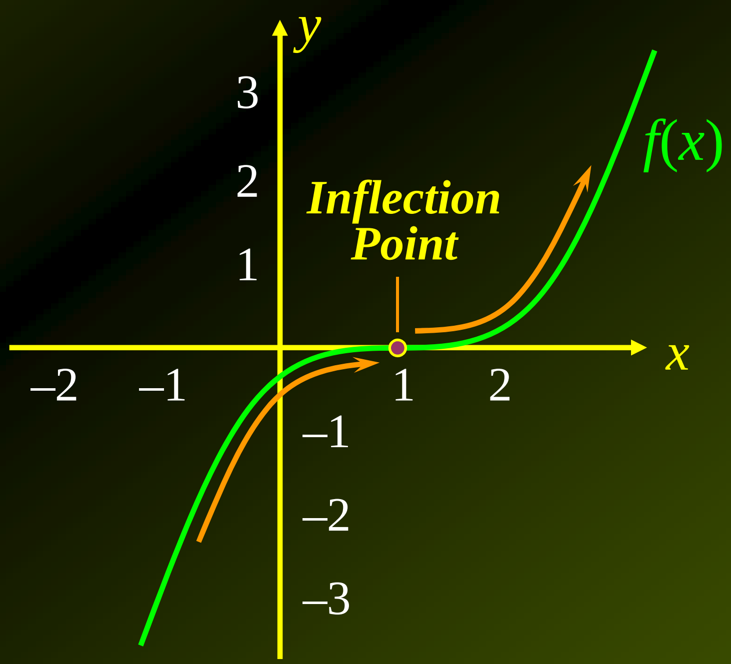
We see that $f''(x) < 0$ when $x < 1$, and $f''(x) > 0$ when $x > 1$.

Example 4 – *Solution*

cont'd

Thus, we find that the graph of f:

- Has one and only inflection point at $f(1) = 0$.
- Is concave downward on the interval $(-\infty, 1)$.
- Is concave upward on the interval $(1, \infty)$.



Applied Example 7 – *Effect of Advertising on Sales*

The total sales *S* (in thousands of dollars) of Arctic Air Co., which makes automobile air conditioners, is related to the amount of money *x* (in thousands of dollars) the company spends on advertising its product by the formula

$$S(x) = -0.01x^3 + 1.5x^2 + 200 \qquad (0 \le x \le 100)$$

Find the inflection point of the function *S*.

Discuss the meaning of this point.

Applied Example 7 – *Solution*

The first two derivatives of S are given by

$$S'(x) = -0.03x^2 + 3x \qquad \text{and} \qquad S''(x) = -0.06x + 3$$

Setting $S''(x) = 0$ gives $x = 50$. So $(50, S(50))$ is the only candidate for an inflection point.

Since $S''(x) > 0$ for $x < 50$, and $S''(x) < 0$ for $x > 50$, the point $(50, 2700)$ is, in fact, an inflection point of S.

This means that the firm experiences diminishing returns on advertising beyond $50,000:

- Every additional dollar spent on advertising increases sales by less than previously spent dollars.

The Second Derivative Test

Maxima occur when a curve is concave downwards, while minima occur when a curve is concave upwards.

This is the basis of the second derivative test:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all the critical numbers of f at which $f'(x) = 0$.
3. Compute $f''(c)$, if it exists, for each critical number c.
    a. If $f''(c) < 0$, then f has a relative maximum at c.
    b. If $f''(c) > 0$, then f has a relative minimum at c.
    c. If $f''(c) = 0$, then the test fails (it is inconclusive).

Example 9

Determine the relative extrema of the function

$$f(x) = x^3 - 3x^2 - 24x + 32$$

Solution:

We have $f'(x) = 3x^2 - 6x - 24 = 3(x-4)(x+2)$ so $f'(x) = 0$ gives the critical numbers $x = -2$ and $x = 4$.

Next, we have $f''(x) = 6x - 6$ which we use to test the critical numbers:

$$f''(-2) = 6(-2) - 6 = -12 - 6 = -18 < 0$$

so, $f(-2) = 60$ is a relative maximum of f.

$$f''(4) = 6(4) - 6 = 24 - 6 = 18 > 0$$

so, $f(4) = -48$ is a relative minimum of f.