4 APPLICATIONS OF THE DERIVATIVE

4.3 Curve Sketching

Vertical Asymptotes

The line $x = a$ is a vertical asymptote of the graph of a function f if either

$$\lim_{x \to a^+} f(x) = \infty \quad \textbf{or} \quad -\infty$$

or

$$\lim_{x \to a^-} f(x) = \infty \quad \textbf{or} \quad -\infty$$

Finding Vertical Asymptotes of Rational Functions

Suppose f is a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomial functions.

Then, the line $x = a$ is a vertical asymptote of the graph of f if $Q(a) = 0$ but $P(a) \neq 0$.

Example 1

Find the vertical asymptotes of the graph of the function

$$f(x) = \frac{x^2}{4 - x^2}$$

Solution:

f is a rational function with $P(x) = x^2$ and $Q(x) = 4 - x^2$.

The zeros of Q are found by solving

$$4 - x^2 = 0$$

$$(2 + x)(2 - x) = 0$$

giving $x = -2$ and $x = 2$.

Example 1 – *Solution*

cont'd

Examine $x = -2$:

$$P(-2) = (-2)^2 = 4 \neq 0,$$

so $x = -2$ is a vertical asymptote.

Examine $x = 2$:

$$P(2) = (2)^2 = 4 \neq 0,$$

so $x = 2$ is also a vertical asymptote.

Horizontal Asymptotes

The line $y = b$ is a horizontal asymptote of the graph of a function f if either

$$\lim_{x \to \infty} f(x) = b \quad \textbf{or} \quad \lim_{x \to -\infty} f(x) = b$$

Example 2

Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{x^2}{4 - x^2}$$

Solution:

We compute

$$\lim_{x\to\infty} \frac{x^2}{4 - x^2} = \lim_{x\to\infty} \frac{1}{\frac{4}{x^2} - 1}$$

$$= \frac{1}{0 - 1}$$

$$= -1$$

and so $y = -1$ is a horizontal asymptote.

Example 2 – *Solution*

cont'd

We compute

$$\lim_{x \to -\infty} \frac{x^2}{4 - x^2} = \lim_{x \to -\infty} \frac{1}{\frac{4}{x^2} - 1}$$

$$= \frac{1}{0 - 1}$$

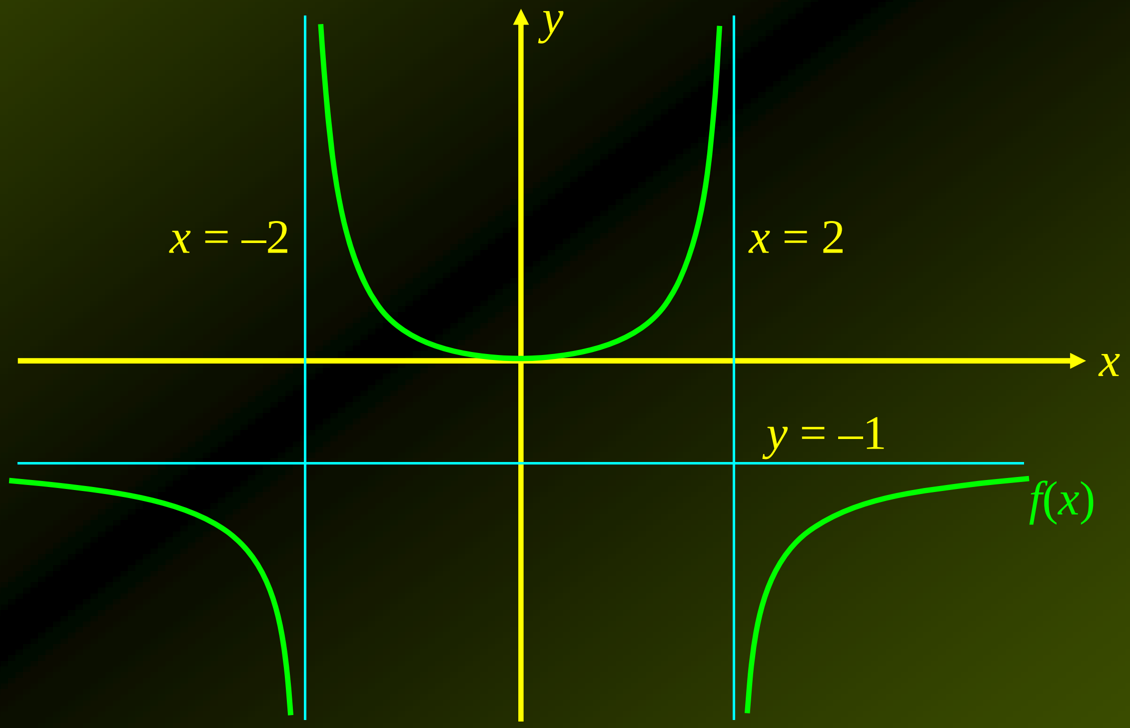
$$= -1$$

also yielding $y = -1$ as a horizontal asymptote.

Example 2 – *Solution*

cont'd

So, the graph of f has two vertical asymptotes $x = -2$ and $x = 2$, and one horizontal asymptote $y = -1$:



Asymptotes and Polynomials

A polynomial function has no vertical asymptotes.

To see this, note that a polynomial function $P(x)$ can be written as a rational function with a denominator equal to 1.

Thus,

$$P(x) = \frac{P(x)}{1}$$

Since the denominator is never zero, P has no vertical asymptotes.

Asymptotes and Polynomials

A polynomial function has no horizontal asymptotes.

If $P(x)$ is a polynomial of degree greater or equal to 1, then

$$\lim_{x \to \infty} P(x) \quad \textbf{and} \quad \lim_{x \to -\infty} P(x)$$

are either infinity or minus infinity; that is, they do not exist.

Therefore, P has no horizontal asymptotes.

A Guide to Sketching a Curve

1. Determine the domain of *f*.
2. Find the *x*- and *y*-intercepts of *f*.
3. Determine the behavior of *f* for large absolute values of *x*.
4. Find all horizontal and vertical asymptotes of *f*.
5. Determine the intervals where *f* is increasing and where *f* is decreasing.
6. Find the relative extrema of *f*.
7. Determine the concavity of *f*.
8. Find the inflection points of *f*.
9. Plot a few additional points to help further identify the shape of the graph of *f* and sketch the graph.

Example 3

Sketch the graph of the function $f(x) = x^3 - 6x^2 + 9x + 2$

Solution:

1. The domain of *f* is the interval $(-\infty, \infty)$.
2. By setting $x = 0$, we find that the *y*-intercept is 2.
   (The *x*-intercept is not readily obtainable)
3. Since

$$\lim_{x \to -\infty} \left(x^3 - 6x^2 + 9x + 2 \right) = -\infty$$

$$\lim_{x \to \infty} \left(x^3 - 6x^2 + 9x + 2 \right) = \infty$$

   we see that *f* decreases without bound as *x* decreases without bound and that *f* increases without bound when *x* increases without bound.

Example 3 – *Solution*

cont'd

4. Since f is a polynomial function, there are no asymptotes.

5. $f'(x) = 3x^2 - 12x + 9 = 3(x - 3)(x - 1)$

   Setting $f'(x) = 0$ gives $x = 1$ and $x = 3$ as critical points.

   Testing with different values of x we find that $f'(x) > 0$ when $x < 1$, $f'(x) < 0$ when $1 < x < 3$, and $f'(x) > 0$ when $x > 3$.

   Thus, f is increasing in the intervals $(-\infty, 1)$ and $(3, \infty)$, and f is decreasing in the interval $(1, 3)$.

Example 3 – *Solution*

cont'd

6. f' changes from positive to negative as we go across $x = 1$, so a relative maximum of f occurs at $(1, f(1)) = (1, 6)$.

   f' changes from negative to positive as we go across $x = 3$, so a relative minimum of f occurs at $(3, f(3)) = (1, 2)$.

7. $f''(x) = 6x - 12 = 6(x - 2)$ which is equal to zero when $x = 2$.

   Testing with different values of x we find that $f''(x) < 0$ when $x < 2$ and $f''(x) > 0$ when $2 < x$.

   Thus, f is concave downward in the interval $(-\infty, 2)$ and concave upward in the interval $(2, \infty)$.

Example 3 – *Solution*

cont'd

8. Since $f''(2) = 0$, we have an inflection point at

$(2, f(2)) = (2, 4)$.

Summarizing, we've found the following:

- Domain: $(-\infty, \infty)$.
- Intercept: $(0, 2)$.
- $\lim_{x \to -\infty} f(x) = -\infty$ **and** $\lim_{x \to \infty} f(x) = \infty$
- Asymptotes: None.
- f is increasing in the intervals $(-\infty, 1)$ and $(3, \infty)$, and f is decreasing in the interval $(1, 3)$.

Example 3 – *Solution*

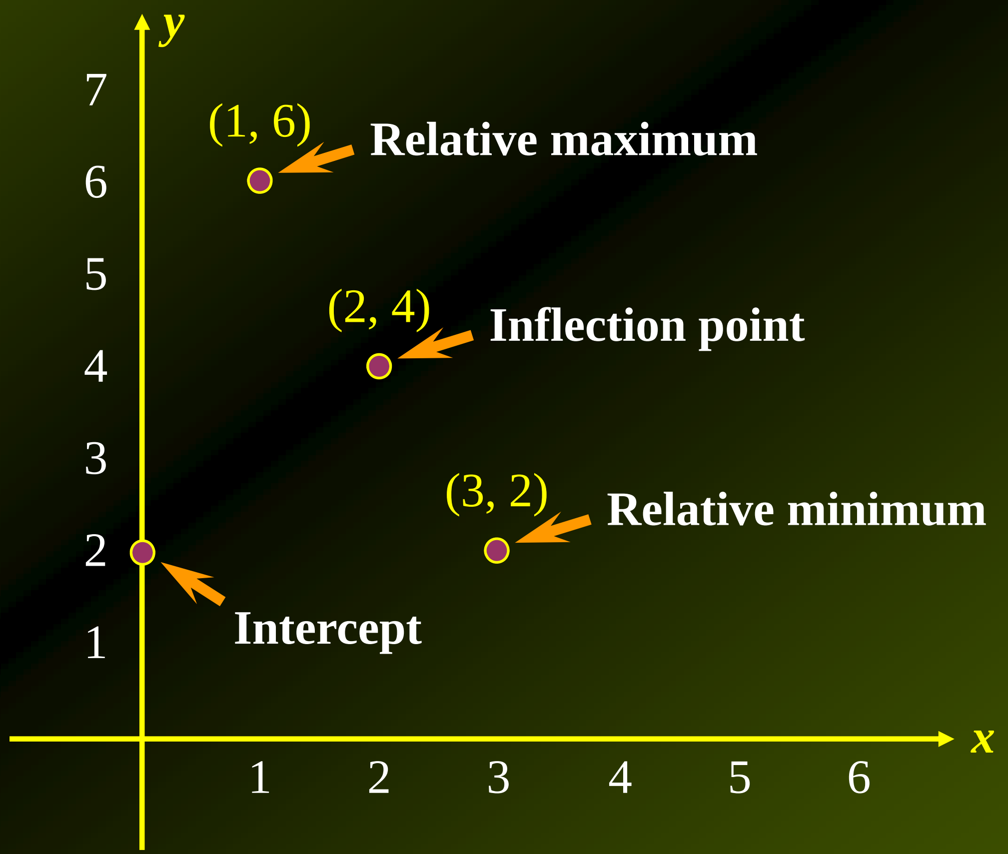
cont'd

- A relative maximum of *f* occurs at (1, 6).
- A relative minimum of *f* occurs at (1, 2).
- *f* is concave downward in the interval $(-\infty, 2)$ and *f* is concave upward in the interval $(2, \infty)$.
- *f* has an inflection point at (2, 4).

Example 3 – *Solution*

cont'd

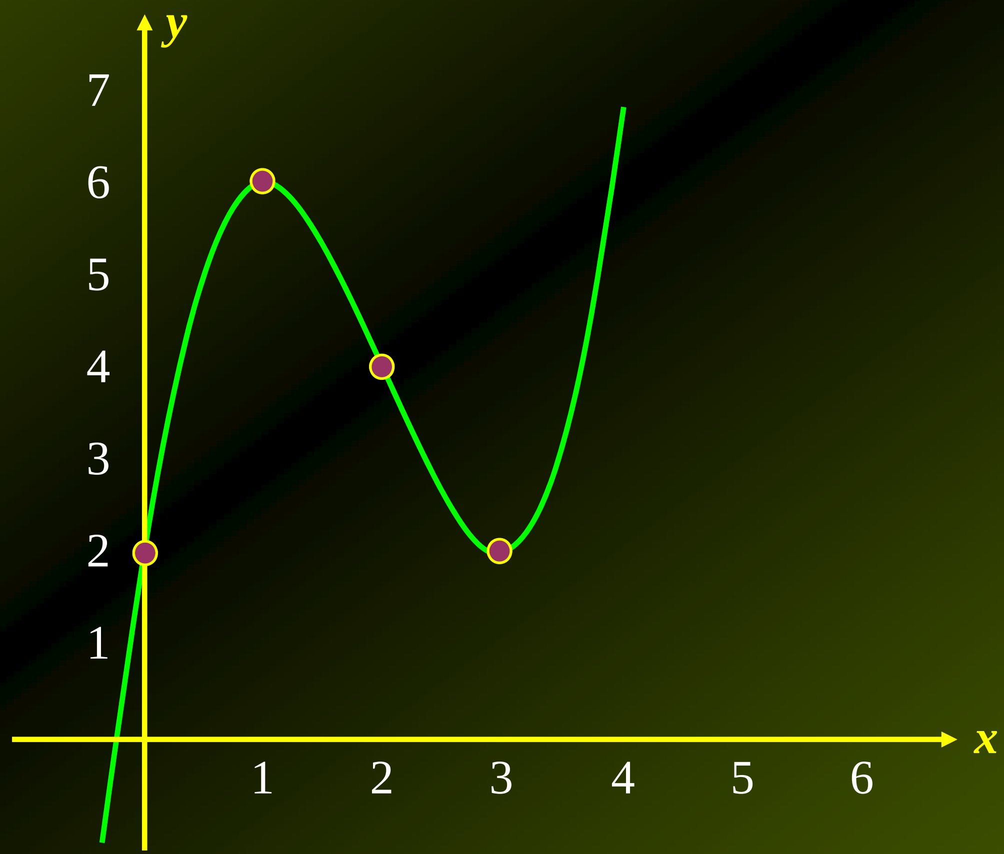
Sketch the graph:



Example 3 – *Solution*

cont'd

Sketch the graph:



Example 4

Sketch the graph of the function $f(x) = \dfrac{x+1}{x-1}$

Solution:

1. *f* is undefined when $x = 1$, so the domain of *f* is the set of all real numbers other than $x = 1$.

2. Setting $y = 0$, gives an *x*-intercept of –1.

   Setting $x = 0$, gives an *y*-intercept of –1.

Example 4 – *Solution*

cont'd

3. Since $\lim_{x \to -\infty} \frac{x+1}{x-1} = 1$ **and** $\lim_{x \to \infty} \frac{x+1}{x-1} = 1$

   we see that *f*(*x*) approaches the line *y* = 1 as |*x*| becomes arbitrarily large.

   - For *x* > 1, *f*(*x*) > 1, so *f* approaches the line *y* = 1 from above.
   - For *x* < 1, *f*(*x*) < 1, so *f* approaches the line *y* = 1 from below.

4. From step three we conclude that *y* = 1 is a horizontal asymptote of *f*.

   Also, the straight line *x* = 1 is a vertical asymptote of *f*.

Example 4 – *Solution*

cont'd

5. $f'(x) = \frac{(x-1)(1)-(x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$

   So, $f'(x)$ is discontinuous at $x = 1$ and is never equal to zero. Testing we find that $f'(x) < 0$ wherever it is defined.

6. From step 5 we see that there are no critical numbers of f, since $f'(x)$ is never equal to zero.

Example 4 – *Solution*

cont'd

7. $f''(x) = \frac{d}{dx}\left[-2(x-1)^{-2}\right] = 4(x-1)^{-3} = \frac{4}{(x-1)^3}$

Testing with different values of x we find that $f''(x) < 0$ when $x < 1$ and $f''(x) > 0$ when $1 < x$.

Thus, f is concave downward in the interval $(-\infty, 1)$ and concave upward in the interval $(1, \infty)$.

8. From point 7 we see there are no inflection points of f, since $f''(x)$ is never equal to zero.

Example 4 – *Solution*

Summarizing, we've found the following:

- Domain: (– ∞, 1) U (1, ∞).
- Intercept: (0, –1); (–1, 0).
- $\lim_{x \to -\infty} f(x) = 1$ **and** $\lim_{x \to \infty} f(x) = 1$
- Asymptotes: *x* = 1 is a vertical asymptote.

  y = 1 is a horizontal asymptote.
- *f* is decreasing everywhere in the domain of *f*.
- Relative extrema: None.
- *f* is concave downward in the interval (– ∞, 1) and *f* is concave upward in the interval (1, ∞).
- *f* has no inflection points.

Example 4 – *Solution*

cont'd

Sketch the graph: