4

APPLICATIONS OF THE DERIVATIVE

4.4 Optimization I

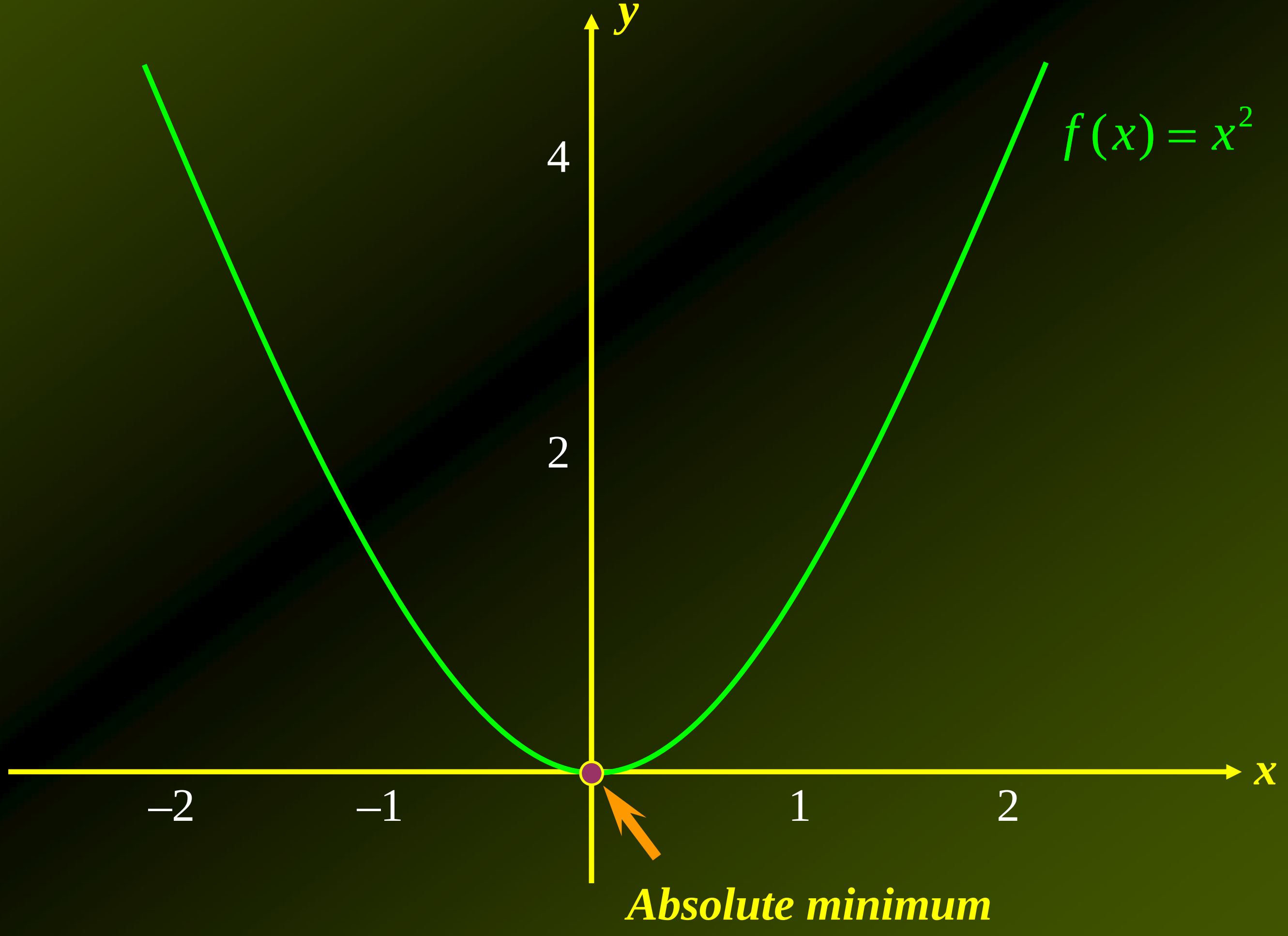
Absolute Extrema

The absolute extrema of a function f

- If $f(x) \leq f(c)$ for all x in the domain of f, then $f(c)$ is called the absolute maximum value of f.
- If $f(x) \geq f(c)$ for all x in the domain of f, then $f(c)$ is called the absolute minimum value of f.

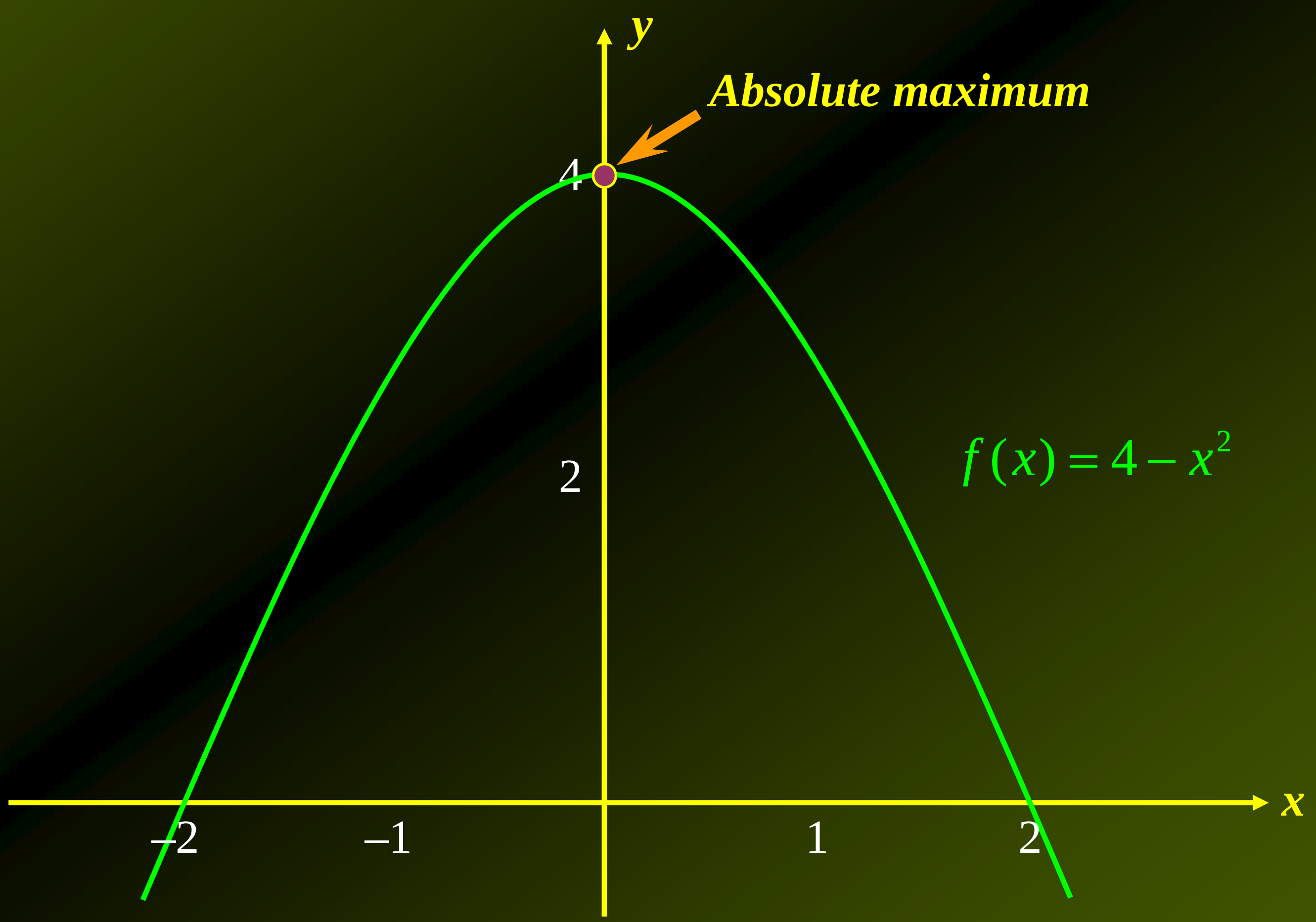
Example

f has an absolute minimum at (0, 0):



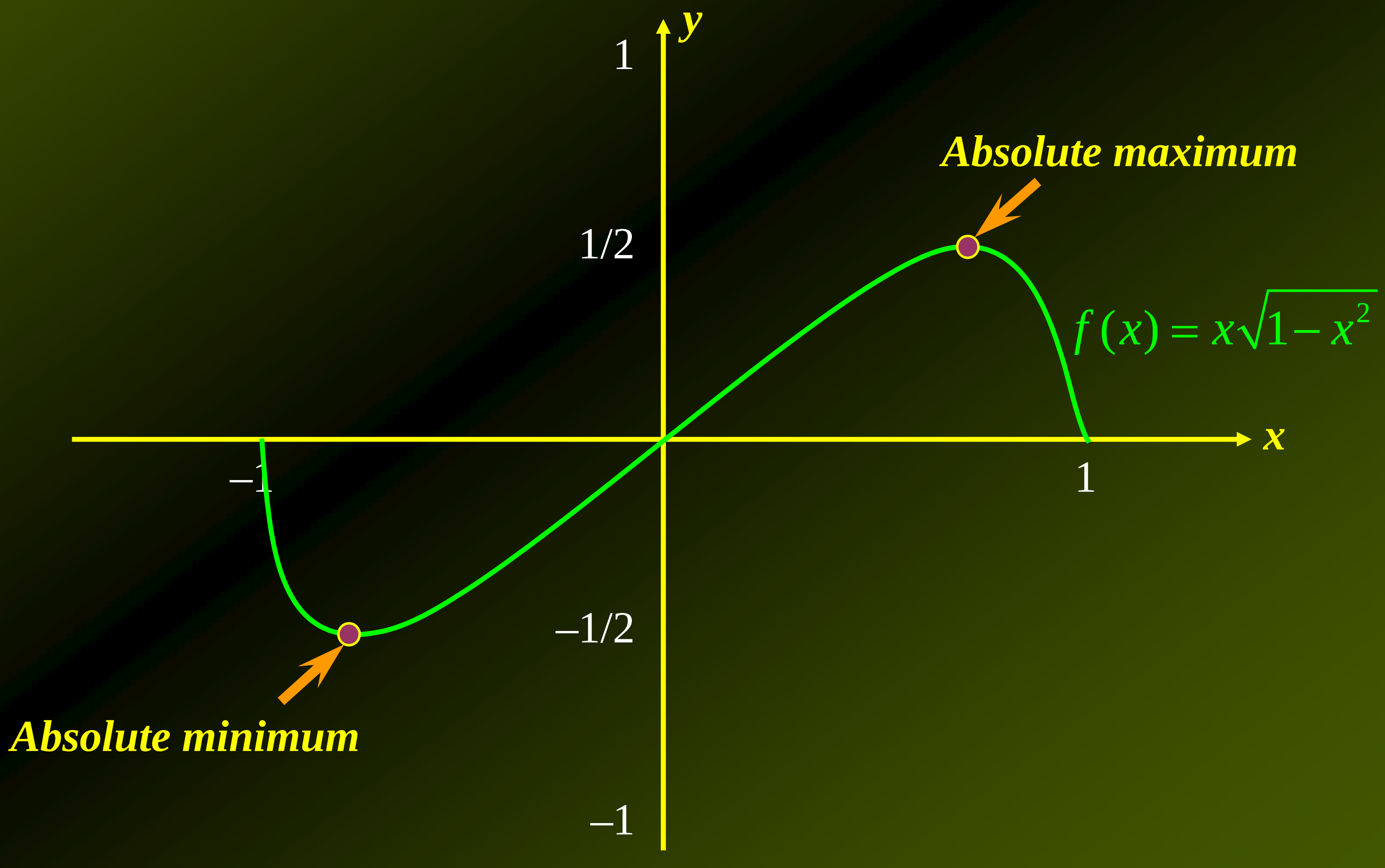
Example

f has an absolute maximum at (0, 4):



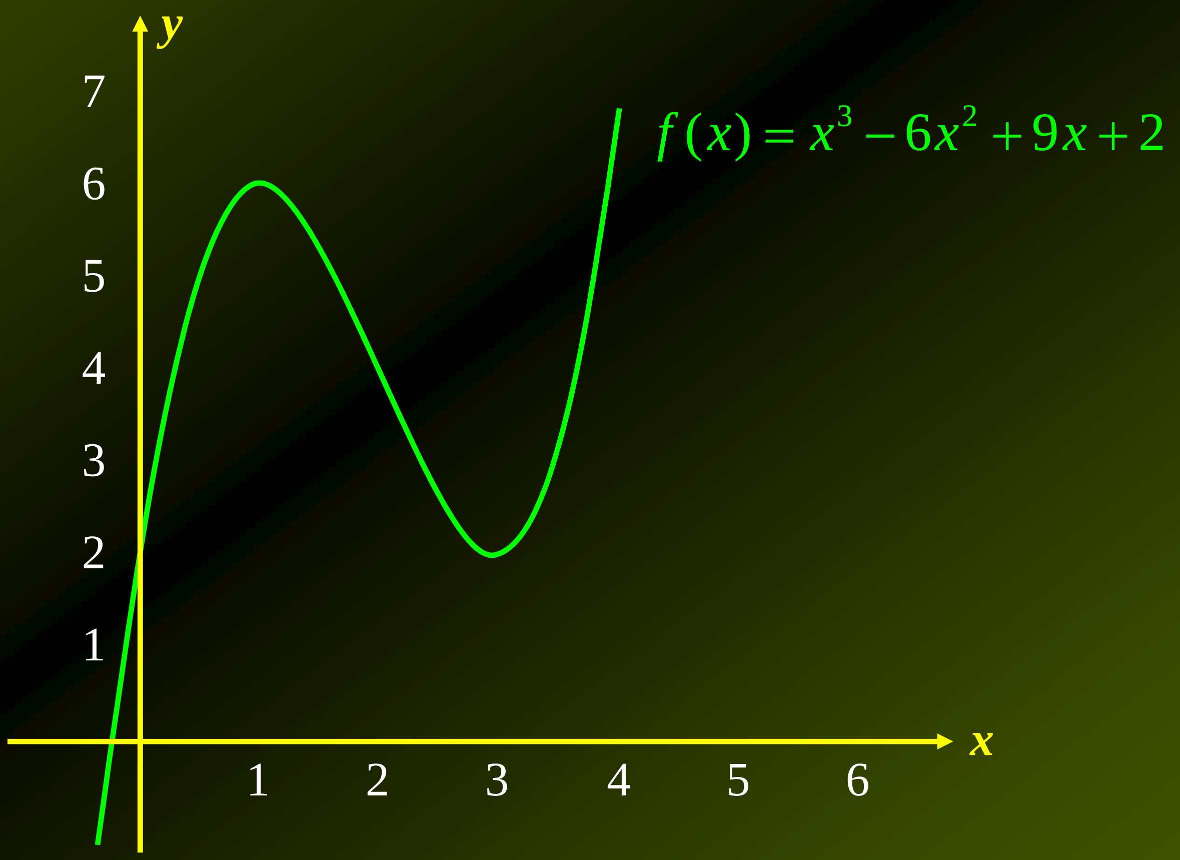
Example

f has an absolute minimum at $(-\sqrt{2}/2, -1/2)$:
and an absolute maximum at $(\sqrt{2}/2, 1/2)$:



Example

f has no absolute extrema:



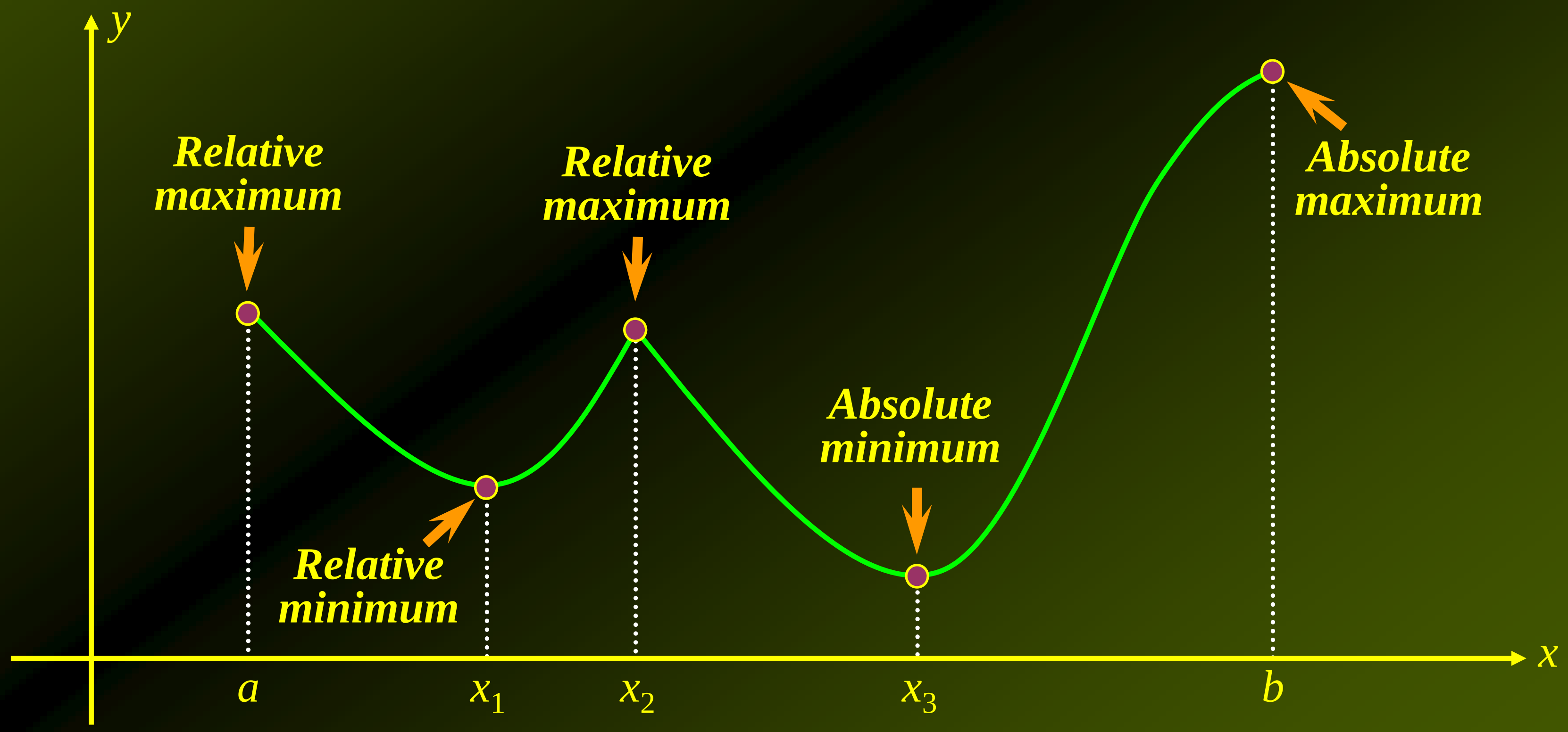
Theorem 3

Absolute Extrema in a Closed Interval

- If a function *f* is continuous on a closed interval [*a*, *b*], then *f* has both an absolute maximum value and an absolute minimum value on [*a*, *b*].

Example

The relative minimum of f at x_3 is also the absolute minimum of f. The right endpoint b of the interval [a, b] gives rise to the absolute maximum value $f(b)$ of f.



Finding Absolute Extrema

To find the absolute extrema of f on a closed interval $[a, b]$.

1. Find the critical numbers of f that lie on (a, b).

2. Compute the value of f at each critical number found in step 1 and compute $f(a)$ and $f(b)$.

3. The absolute maximum value and absolute minimum value of f will correspond to the largest and smallest numbers, respectively, found in step 2.

Example 1

Find the absolute extrema of the function $F(x) = x^2$ defined on the interval $[-1, 2]$.

Solution:

The function F is continuous on the closed interval $[-1, 2]$ and differentiable on the open interval $(-1, 2)$.

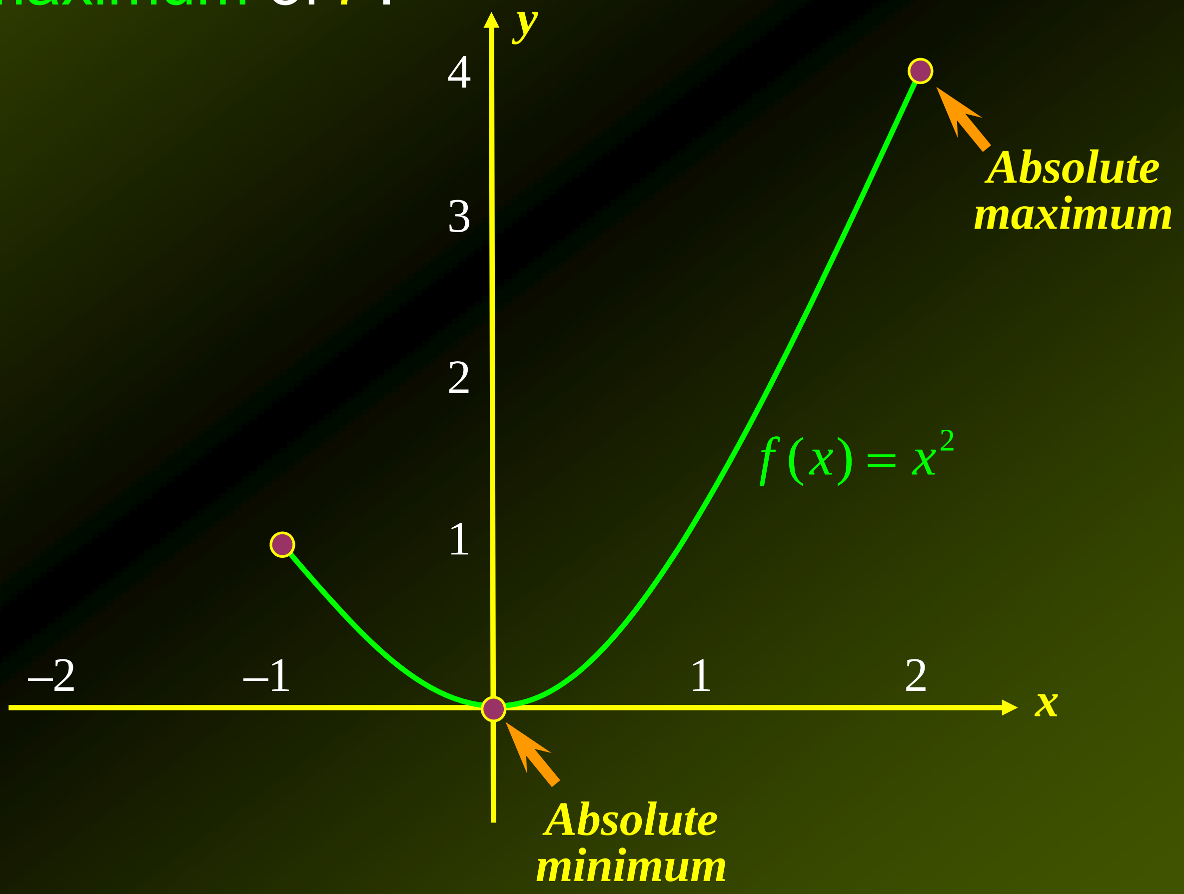
Setting $F' = 0$, we get $F'(x) = 2x = 0$, so there is only one critical point at $x = 0$.

Example 1 – *Solution*

cont'd

So, $F(-1) = (-1)^2 = 1$, $F(0) = (0)^2 = 0$, and $F(2) = (2)^2 = 4$.

It follows that 0 is the absolute minimum of F, and 4 is the absolute maximum of F.



Example 2

Find the absolute extrema of the function

$$f(x) = x^3 - 2x^2 - 4x + 4$$

defined on the interval [0, 3].

Solution:

The function f is continuous on the closed interval [0, 3] and differentiable on the open interval (0, 3).

Setting $f' = 0$, we get

$$f'(x) = 3x^2 - 4x - 4 = (3x+2)(x-2) = 0$$

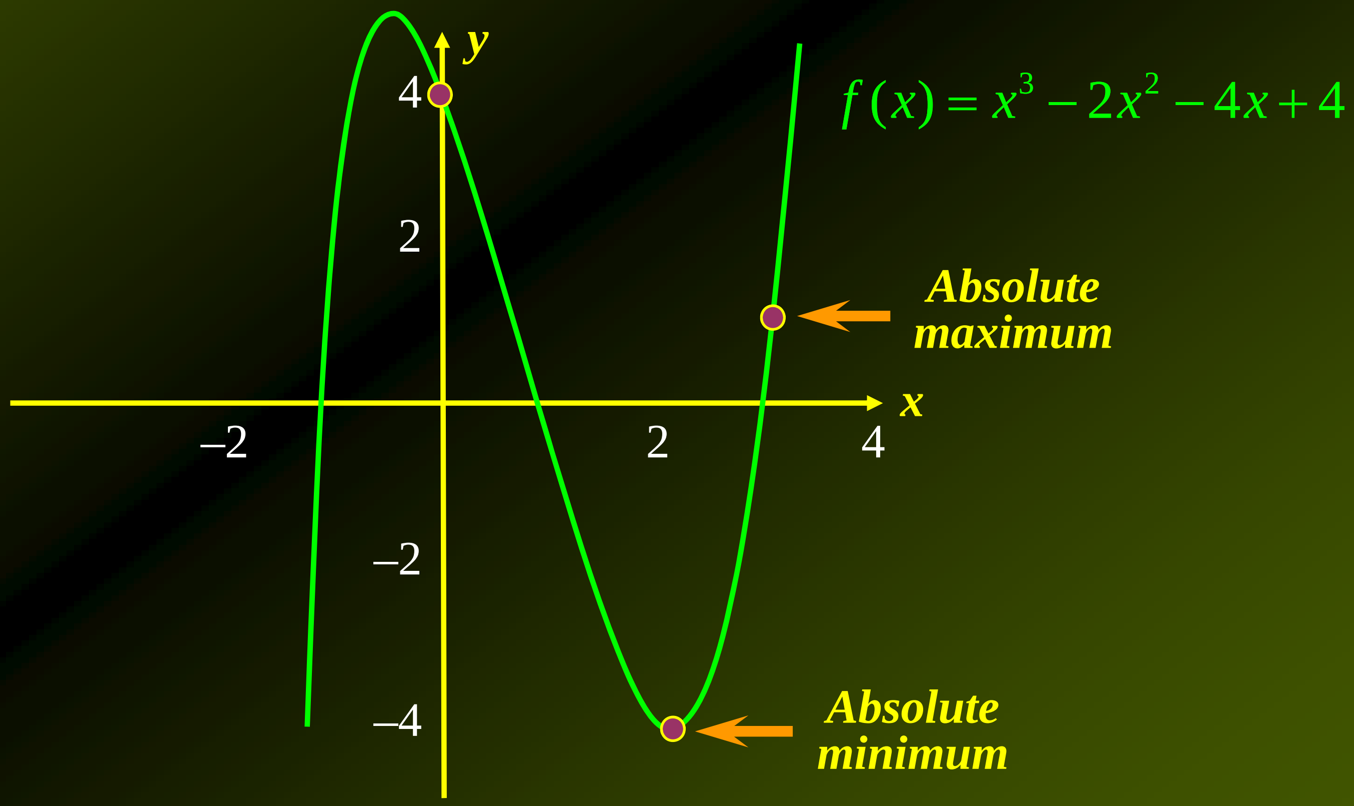
which gives two critical points at $x = -2/3$ and $x = 2$.

Example 2 – *Solution*

cont'd

We drop $x = -2/3$ since it lies outside the interval [0, 3].

So, $f(0) = 4$, $f(2) = -4$, and $f(3) = 1$. It follows that -4 is the absolute minimum of f, and 4 is the absolute maximum of f.



Applied Example 4 – *Maximizing Profits*

Acrosonic's total profit (in dollars) from manufacturing and selling *x* units of their model F speakers is given by

$$P(x) = -0.02x^2 + 300x - 200{,}000 \qquad (0 \leq x \leq 20{,}000)$$

How many units of the loudspeaker system must Acrosonic produce to maximize profits?

Solution:

To find the absolute maximum of *P* on [0, 20,000], first find the stationary points of *P* on the interval (0, 20,000).

Applied Example 4 – *Solution*

cont'd

Setting f' = 0, we get

$$P'(x) = -0.04x + 300 = 0$$

$$x = \frac{300}{0.04} = 7500$$

which gives us only one stationary point at $x = 7500$.

Evaluating the only stationary point we get

$$P(7500) = 925{,}000$$

Evaluating the endpoints we get

$$P(0) = -200{,}000$$
$$P(20{,}000) = -2{,}200{,}000$$

Applied Example 4 – *Solution*

cont'd

Thus, Acrosonic will realize the maximum profit of $925,000 by producing 7500 speakers.