4 APPLICATIONS OF THE DERIVATIVE

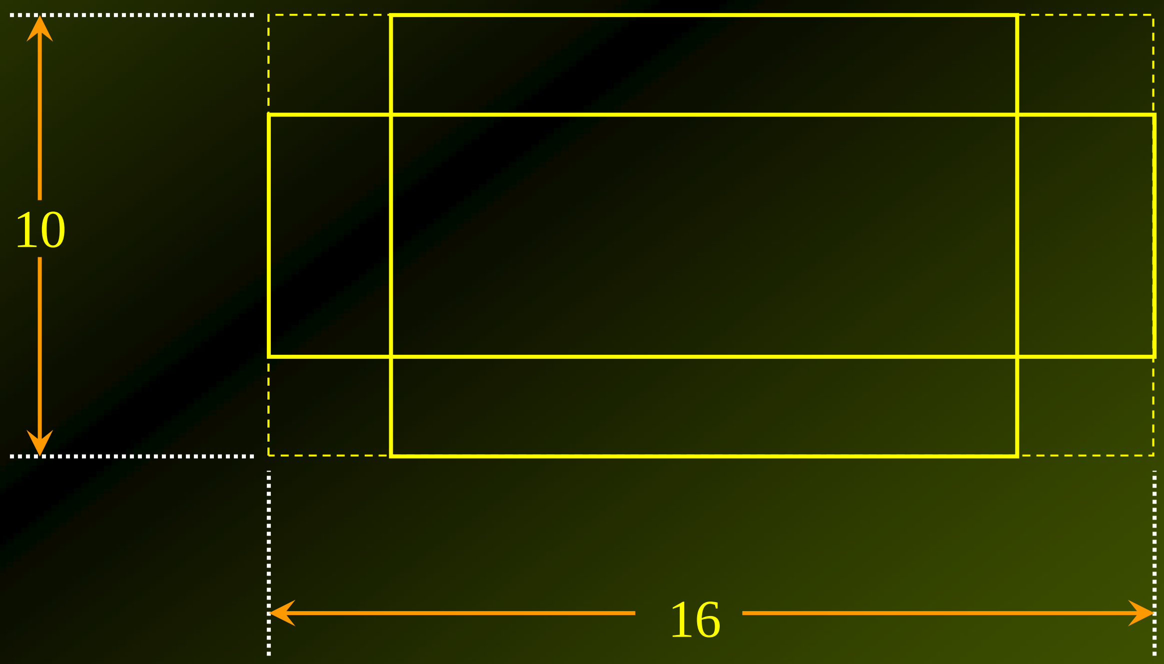
4.5 Optimization II

Guidelines for Solving Optimization Problems

1. Assign a letter to each variable mentioned in the problem. If appropriate, draw and label a figure.
2. Find an expression for the quantity to be optimized.
3. Use the conditions given in the problem to write the quantity to be optimized as a function *f* of one variable. Note any restrictions to be placed on the domain of *f* from physical considerations of the problem.
4. Optimize the function *f* over its domain using the methods of Section 4.4.

Applied Maximization Problem 2 – *Packaging*

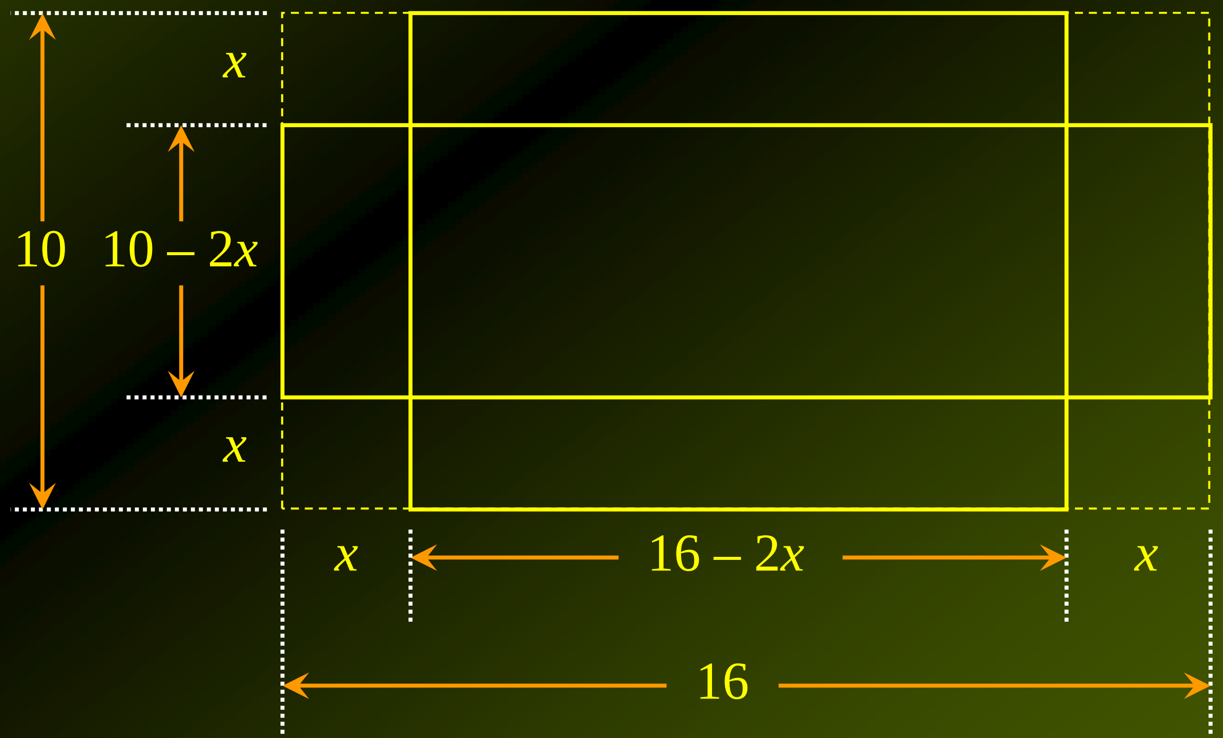
By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16 inches long and 10 inches wide, find the dimensions of the box that will yield the maximum volume.



Applied Maximization Problem 2 – *Solution*

1. Let x denote the length in inches of one side of each of the identical squares to be cut out of the cardboard.

   The dimensions of the box are $(16 - 2x)$ by $(10 - 2x)$ by x in.



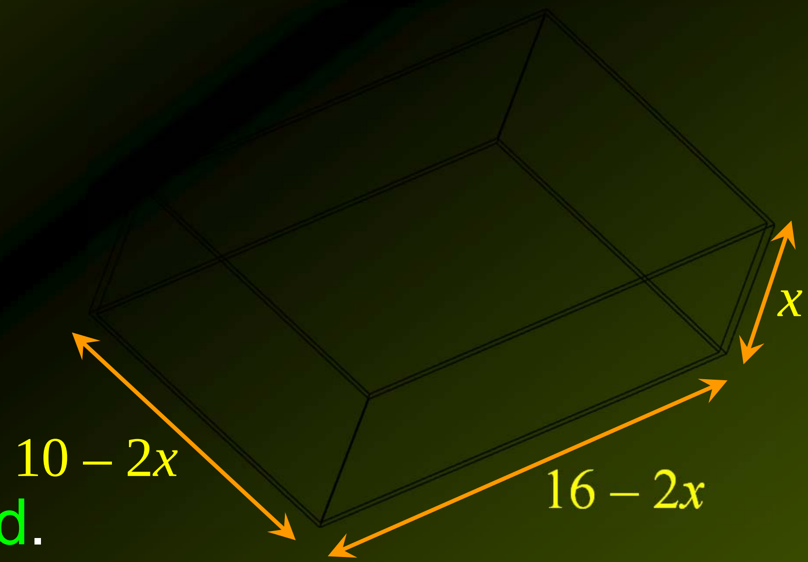
Applied Maximization Problem 2 – *Solution* cont'd

2. Let *V* denote the volume (in cubic inches) of the resulting box.

The volume,

$$V = (16 - 2x)(10 - 2x)x$$

$$= 4(x^3 - 13x^2 + 40x)$$



is the quantity to be maximized.

Applied Maximization Problem 2 – *Solution* cont'd

3. Each side of the box must be nonnegative, so x must satisfy the inequalities $x \geq 0$, $16 - 2x \geq 0$, and $10 - 2x \geq 0$.

   All these inequalities are satisfied if $0 \leq x \leq 5$.

   Therefore, the problem at hand is equivalent to finding the absolute maximum of

$$V = f(x) = 4(x^3 - 13x^2 + 40x)$$

   on the closed interval [0, 5].

Applied Maximization Problem 2 – *Solution* cont'd

4. f is continuous on [0, 5]. Setting $f'(x) = 0$ we get

$$f'(x) = 4(3x^2 - 26x + 40)$$
$$= 4(3x - 20)(x - 2)$$

which yields the critical numbers $x = 20/3$ and $x = 2$.

We discard $x = 20/3$ for being outside the interval [0, 5].

We evaluate f at the critical point and at the endpoints:

$f(0) = 0$ $\quad$ $f(2) = 144$ $\quad$ $f(5) = 0$

Thus, the volume of the box is maximized by taking $x = 2$.

The resulting dimensions of the box are $12'' \times 6'' \times 2''$.

Applied Minimization Problem 5 – *Inventory Control*

Dixie's Import-Export is the sole seller of the Excalibur 250 cc motorcycle. Management estimates that the demand for these motorcycles will be 10,000 for the coming year and that they will sell at a uniform rate throughout the year. The cost incurred in ordering each shipment of motorcycles is $10,000, and the cost per year of storing each motorcycle is $200.

Applied Minimization Problem 5 – *Inventory Control* cont'd

Dixie's management faces the following problem:

- Ordering too many motorcycles at one time increases storage cost.
- On the other hand, ordering too frequently increases the ordering costs.

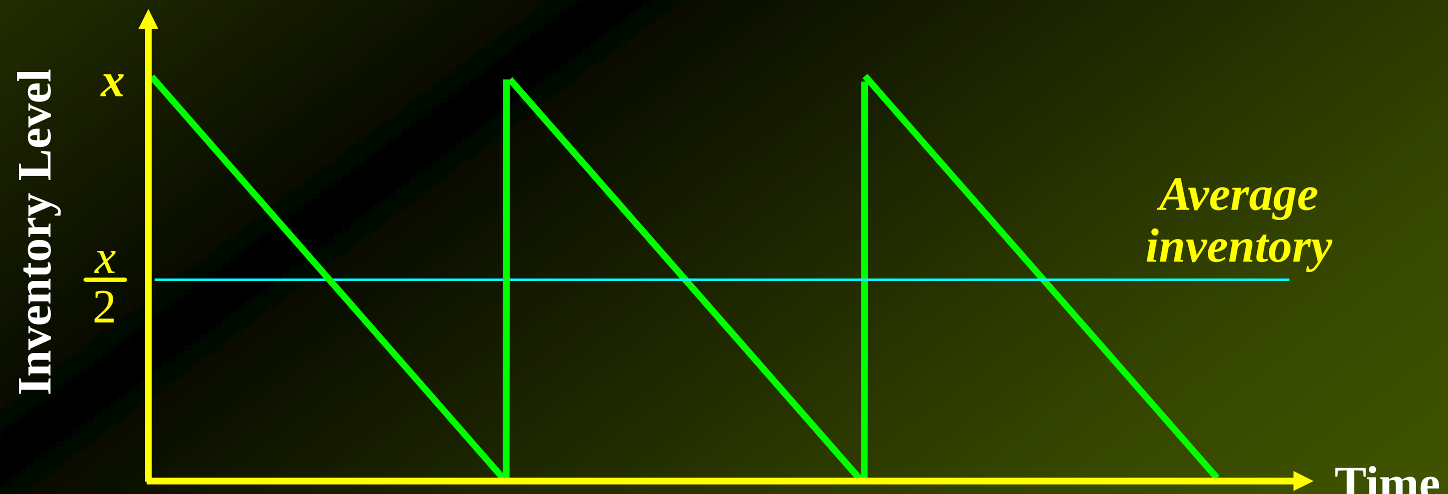
How large should each order be, and how often should orders be placed, to minimize ordering and storage costs?

Applied Minimization Problem 5 – *Solution*

Let x denote the number of motorcycles in each order.

Assuming each shipment arrives just as the previous shipment is sold out, the average number of motorcycles in storage during the year is $x/2$, as shown below:

Thus, Dixie's storage cost for the year is given by $200(x/2)$, or $100x$ dollars.



Applied Minimization Problem 5 – *Solution* cont'd

Next, since the company requires 10,000 motorcycles for the year and since each order is for x motorcycles, the number of orders required is

$$\frac{10,000}{x}$$

This gives an ordering cost of

$$10,000\left(\frac{10,000}{x}\right)=\frac{100,000,000}{x}$$

dollars for the year.

Thus, the total yearly cost incurred by Dixie's, including ordering and storage costs, is given by

$$C(x)=100x+\frac{100,000,000}{x}$$

Applied Minimization Problem 5 – *Solution* cont'd

The problem is reduced to finding the absolute minimum of the function C in the interval (0,10,000].

To accomplish this, we compute

$$C'(x) = 100 - \frac{100,000,000}{x^2}$$

Setting $C'(x) = 0$ and solving we obtain $x = \pm 1000$.

We reject the negative for being outside the domain. So we have $x = 1000$ as the only critical number.

Applied Minimization Problem 5 – *Solution* cont'd

Now we find

$$C''(x) = \frac{200{,}000{,}000}{x^3}$$

Since $C''(1000) > 0$, the second derivative test implies that $x = 1000$ is a relative minimum of C.

Also, since $C''(1000) > 0$ for all x in $(0,10{,}000]$, the function C is concave upward everywhere so that also gives the absolute minimum of C.

Thus, to minimize the ordering and storage costs, Dixie's should place 10,000/1000, or 10, orders per year, each for a shipment of 1000 motorcycles.