5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.1 Exponential Functions

Exponential Function

The function defined by

$$f(x) = b^x \qquad (b > 0,\ b \neq 1)$$

is called an exponential function with base *b* and exponent *x*.

The domain of *f* is the set of all real numbers.

Example

The exponential function with base 2 is the function

$$f(x) = 2^x$$

with domain $(-\infty, \infty)$.

Find the values of $f(x)$ for selected values of x follow:

$$f(3) = 2^3 = 8$$

$$f\left(\frac{3}{2}\right) = 2^{3/2} = 2 \cdot 2^{1/2} = 2\sqrt{2}$$

$$f(0) = 2^0 = 1$$

Example

$$f(-1) = 2^{-1} = \frac{1}{2}$$

$$f\left(-\frac{2}{3}\right) = 2^{-2/3} = \frac{1}{2^{2/3}} = \frac{1}{\sqrt[3]{4}}$$

Laws of Exponents

Let *a* and *b* be positive numbers and let *x* and *y* be real numbers. Then,

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $\left(b^x\right)^y = b^{xy}$
4. $\left(ab\right)^x = a^x b^x$
5. $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Example 2

Let $f(x) = 2^{2x-1}$. Find the value of x for which $f(x) = 16$.

Solution:

We want to solve the equation

$$2^{2x-1} = 16 \quad = 2^4$$

But this equation holds if and only if

$$2x - 1 = 4$$

giving $x = \frac{5}{2}$.

Example 3

Sketch the graph of the exponential function $f(x) = 2^x$.

Solution:

First, recall that the domain of this function is the set of real numbers.

Next, putting $x = 0$ gives $y = 2^0 = 1$, which is the y-intercept. (There is no x-intercept, since there is no value of x for which $y = 0$.)

Example 3 – *Solution*

cont'd

Now, consider a few values for x:

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	1/32	1/16	1/8	1/4	1/2	1	2	4	8	16	32

Note that 2^x approaches zero as x decreases without bound: There is a horizontal asymptote at $y = 0$.

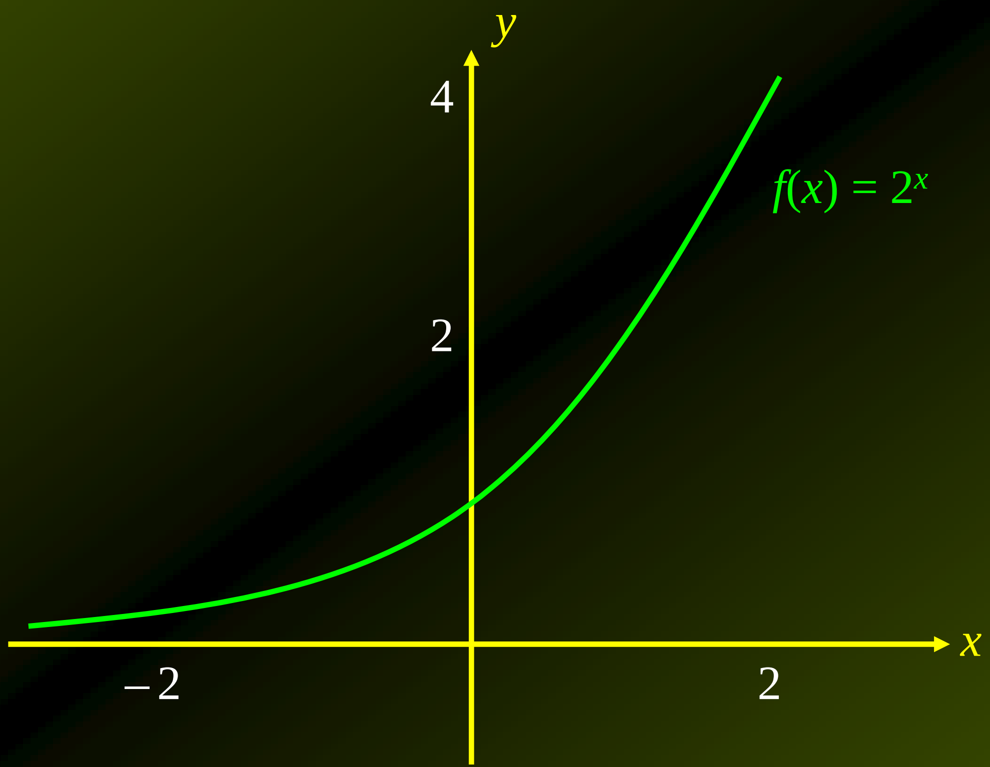
Furthermore, 2^x increases without bound when x increases without bound.

Thus, the range of f is the interval $(0, \infty)$.

Example 3 – *Solution*

cont'd

Finally, sketch the graph:



Example 4

Sketch the graph of the exponential function $f(x) = (1/2)^x$.

Solution:

First, recall again that the domain of this function is the set of real numbers.

Next, putting $x = 0$ gives $y = (1/2)^0 = 1$, which is the y-intercept.

(There is no x-intercept, since there is no value of x for which $y = 0$.)

Example 4 – *Solution*

cont'd

Now, consider a few values for x:

x	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5
y	32	16	8	4	2	1	1/2	1/4	1/8	1/16	1/32

Note that $(1/2)^x$ increases without bound when x decreases without bound.

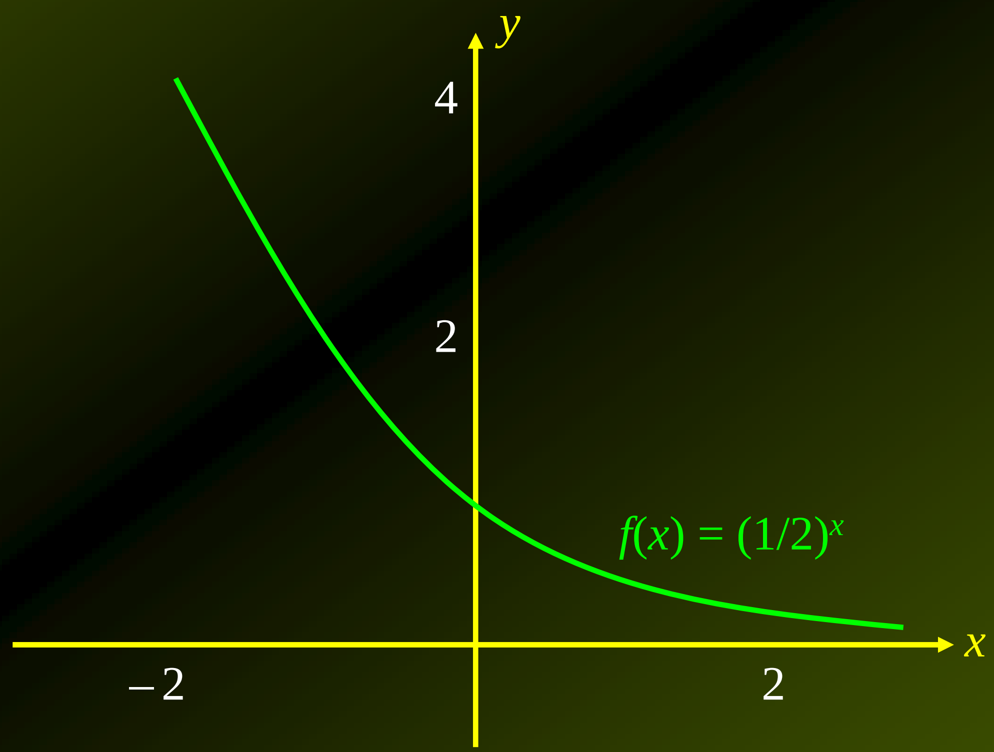
Furthermore, $(1/2)^x$ approaches zero as x increases without bound: there is a horizontal asymptote at $y = 0$.

As before, the range of f is the interval $(0, \infty)$.

Example 4 – *Solution*

cont'd

Finally, sketch the graph:



Example 4 – *Solution*

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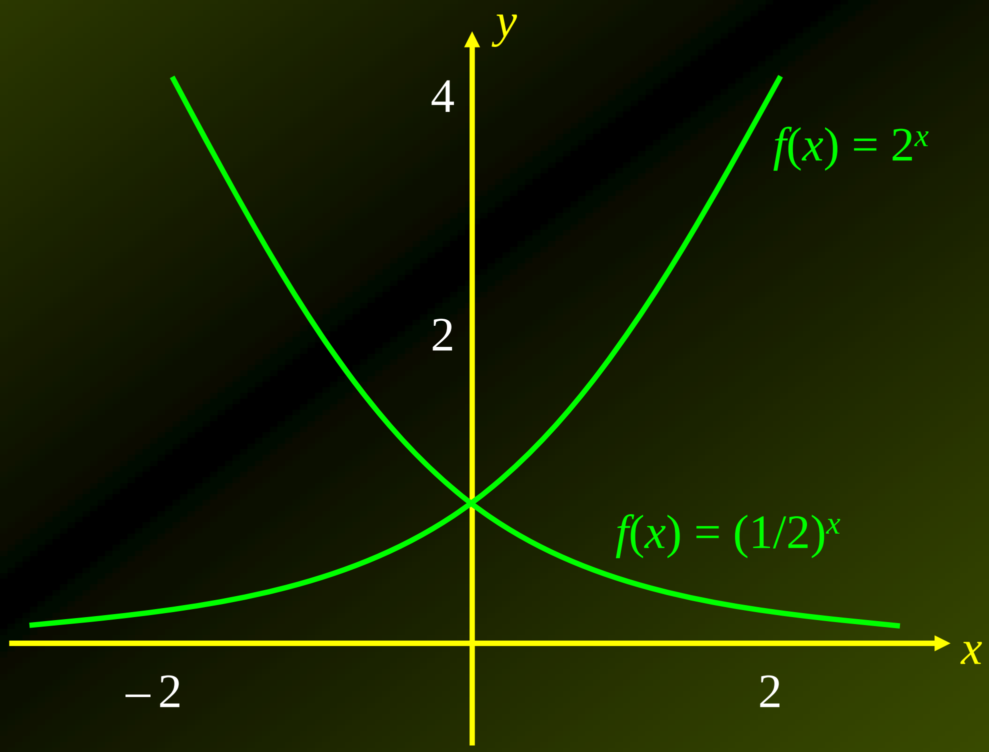
Note the symmetry between the two functions:



Properties of Exponential Functions

The exponential function $y = b^x$ $(b > 0, b \neq 1)$ has the following properties:

1. Its domain is $(-\infty, \infty)$.
2. Its range is $(0, \infty)$.
3. Its graph passes through the point $(0, 1)$.
4. It is continuous on $(-\infty, \infty)$.
5. It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.

The Base *e*

Exponential functions to the base *e*, where *e* is an irrational number whose value is 2.7182818…, play an important role in both theoretical and applied problems.

It can be shown that

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m$$

Example 5

Sketch the graph of the exponential function $f(x) = e^x$.

Solution:
Since $e^x > 0$ it follows that the graph of $y = e^x$ is similar to the graph of $y = 2^x$.

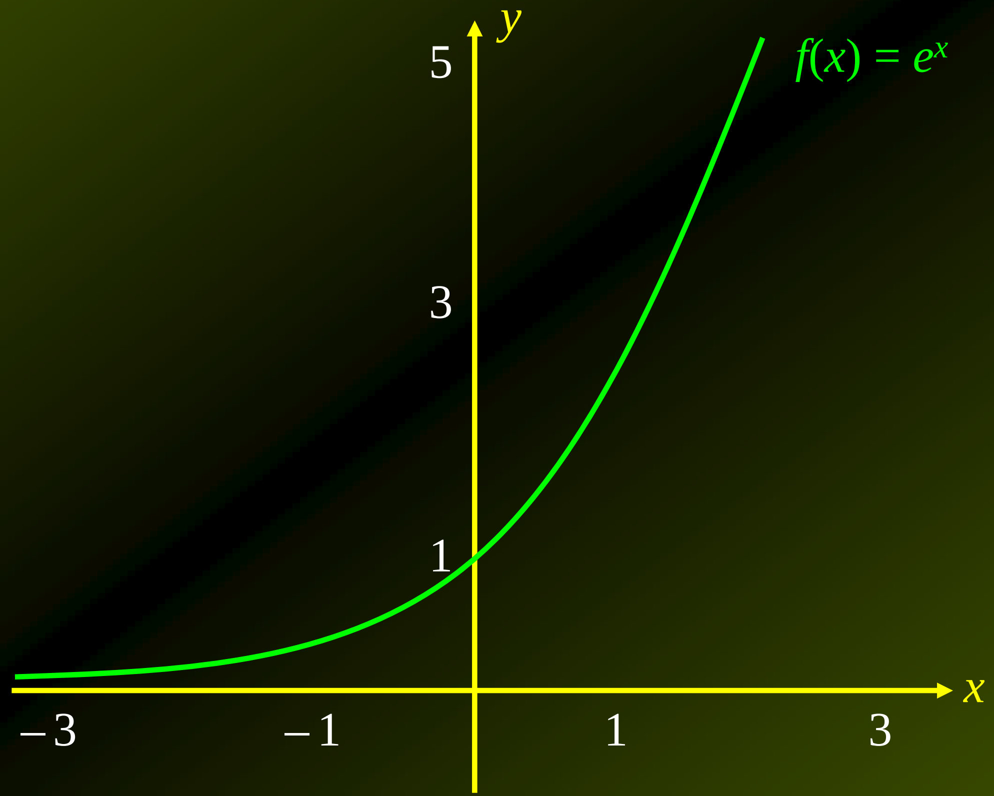
Consider a few values for x:

x	–3	–2	–1	0	1	2	3
y	0.05	0.14	0.37	1	2.72	7.39	20.09

Example 5 – *Solution*

cont'd

Sketching the graph:



Example 6

Sketch the graph of the exponential function $f(x) = e^{-x}$.

Solution:

Since $e^{-x} > 0$ it follows that $0 < 1/e < 1$ and so $f(x) = e^{-x} = 1/e^x = (1/e)^x$ is an exponential function with base less than 1.

Therefore, it has a graph similar to that of $y = (1/2)^x$.

Consider a few values for x:

x	–3	–2	–1	0	1	2	3
y	20.09	7.39	2.72	1	0.37	0.14	0.05

Example 6 – *Solution*

cont'd

Sketching the graph: