5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.2 Logarithmic Functions

Logarithms

We've discussed exponential equations of the form

$$y = b^x \qquad (b > 0,\ b \neq 1)$$

But what about solving the same equation for y?

You may recall that y is called the logarithm of x to the base b, and is denoted $\log_b x$.

Logarithm of x to the base b

$y = \log_b x$ if and only if $x = b^y$ $\quad (x > 0)$

Example 2(a)

Solve $\log_3 x = 4$ for x:

Solution:

By definition, $\log_3 x = 4$ implies $x = 3^4 = 81$.

Example 2(b)

Solve $\log_{16} 4 = x$ for x:

Solution:

$\log_{16} 4 = x$ is equivalent to $4 = 16^x = (4^2)^x = 4^{2x}$, or $4^1 = 4^{2x}$, from which we deduce that

$$2x = 1$$

$$x = \frac{1}{2}$$

Example 2(c)

Solve $\log_x 8 = 3$ for x:

Solution:

By definition, we see that $\log_x 8 = 3$ is equivalent to

$$8 = 2^3 = x^3$$

$$x = 2$$

Logarithmic Notation

$\log x = \log_{10} x$ Common logarithm

$\ln x = \log_e x$ Natural logarithm

Laws of Logarithms

If *m* and *n* are positive numbers, then

1. $\log_b mn = \log_b m + \log_b n$
2. $\log_b \dfrac{m}{n} = \log_b m - \log_b n$
3. $\log_b m^n = n \log_b m$
4. $\log_b 1 = 0$
5. $\log_b b = 1$

Example 4(a)

Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find $\log 15$.

Solution:

$$\begin{aligned}\log 15 &= \log 3 \cdot 5 \\ &= \log 3 + \log 5 \\ &\approx 0.4771 + 0.6990 \\ &= 1.1761\end{aligned}$$

Example 4(b)

Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find $\log 7.5$.

Solution:

$$\log 7.5 = \log(15/2)$$

$$= \log(3 \cdot 5/2)$$

$$= \log 3 + \log 5 - \log 2$$

$$\approx 0.4771 + 0.6990 - 0.3010$$

$$= 0.8751$$

Example 4(c)

Given that log 2 ≈ 0.3010, log 3 ≈ 0.4771, and log 5 ≈ 0.6990, use the laws of logarithms to find $\log 81$.

Solution:

$$\log 81 = \log 3^4$$

$$= 4\log 3$$

$$\approx 4(0.4771)$$

$$= 1.9084$$

Example 4(d)

Given that $\log 2 \approx 0.3010$, $\log 3 \approx 0.4771$, and $\log 5 \approx 0.6990$, use the laws of logarithms to find $\log 50$.

Solution:

$$\log 50 = \log 5 \cdot 10$$

$$= \log 5 + \log 10$$

$$\approx 0.6990 + 1$$

$$= 1.6990$$

Example 5(a)

Expand and simplify the expression $\log_3 x^2 y^3$.

Solution:

$$\log_3 x^2 y^3 = \log_3 x^2 + \log_3 y^3$$
$$= 2\log_3 x + 3\log_3 y$$

Example 5(b)

Expand and simplify the expression $\log_2 \dfrac{x^2+1}{2^x}$.

Solution:

$$\log_2 \frac{x^2+1}{2^x} = \log_2\left(x^2+1\right) - \log_2 2^x$$

$$= \log_2\left(x^2+1\right) - x\log_2 2$$

$$= \log_2\left(x^2+1\right) - x$$

Example 5(c)

Expand and simplify the expression $\ln \frac{x^2\sqrt{x^2-1}}{e^x}$.

Solution:

$$\ln \frac{x^2\sqrt{x^2-1}}{e^x} = \ln \frac{x^2(x^2-1)^{1/2}}{e^x}$$

$$= \ln x^2 + \ln(x^2-1)^{1/2} - \ln e^x$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x\ln e$$

$$= 2\ln x + \frac{1}{2}\ln(x^2-1) - x$$

Logarithmic Function

The function defined by

$$f(x) = \log_b x \qquad (b > 0),\ b \neq 1)$$

is called the logarithmic function with base *b*.

The domain of *f* is the set of all positive numbers.

Properties of Logarithmic Functions

The logarithmic function

$$y = \log_b x \qquad (b > 0, b \neq 1)$$

has the following properties:

1. Its domain is $(0, \infty)$.
2. Its range is $(-\infty, \infty)$.
3. Its graph passes through the point $(1, 0)$.
4. It is continuous on $(0, \infty)$.
5. It is increasing on $(0, \infty)$ if $b > 1$and decreasing on $(0, \infty)$ if $b < 1$.

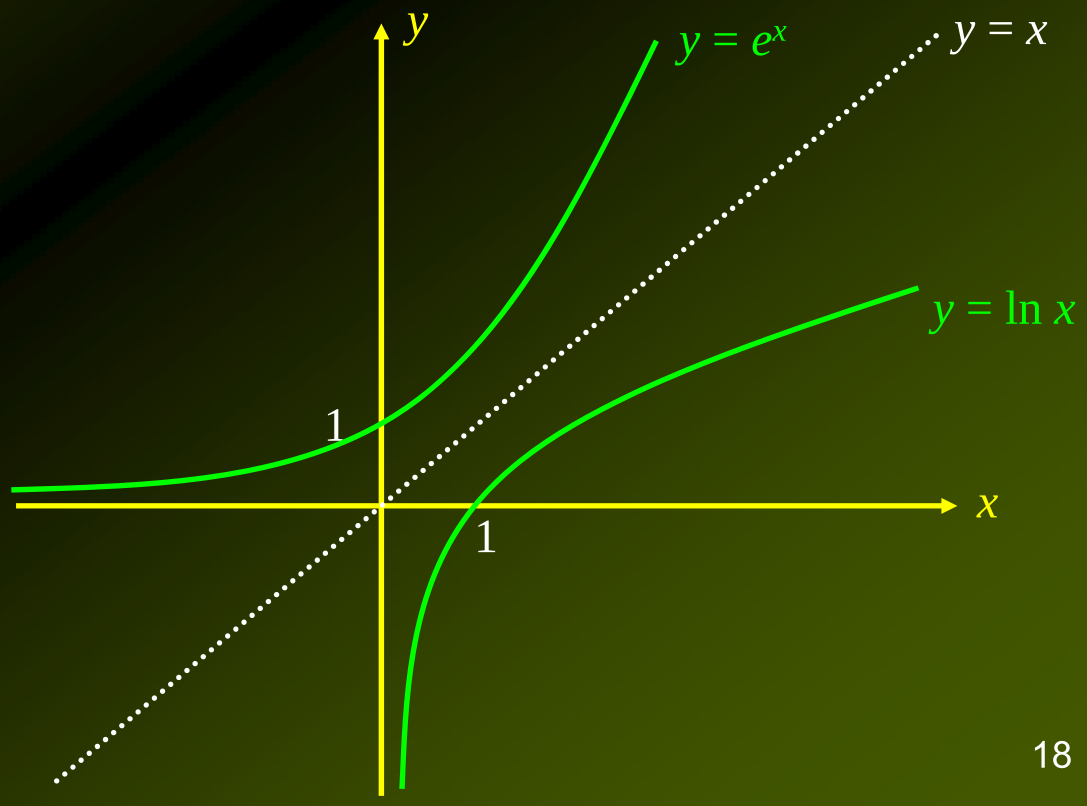
Example 6

Sketch the graph of the function $y = \ln x$.

Solution:

We first sketch the graph of $y = e^x$.

The required graph is the mirror image of the graph of $y = ex$ with respect to the line $y = x$:



Properties Relating Exponential and Logarithmic Functions

Properties relating e^x and $\ln x$:

$e^{\ln x} = x$ $(x > 0)$

$\ln e^x = x$ (for any real number x)

Example 7

Solve the equation $2e^{x+2} = 5$.

Solution:

Divide both sides of the equation by 2 to obtain:

$$e^{x+2} = \frac{5}{2} = 2.5$$

Take the natural logarithm of each side of the equation and solve:

$$\ln e^{x+2} = \ln 2.5$$

$$(x+2)\ln e = \ln 2.5$$

$$x+2 = \ln 2.5$$

$$x = -2 + \ln 2.5$$

$$x \approx -1.08$$

Example 8

Solve the equation $5 \ln x + 3 = 0$.

Solution:

Add – 3 to both sides of the equation and then divide both sides of the equation by 5 to obtain:

$$5 \ln x = -3$$

$$\ln x = -\frac{3}{5} = -0.6$$

and so:

$$e^{\ln x} = e^{-0.6}$$

$$x = e^{-0.6}$$

$$x \approx 0.55$$