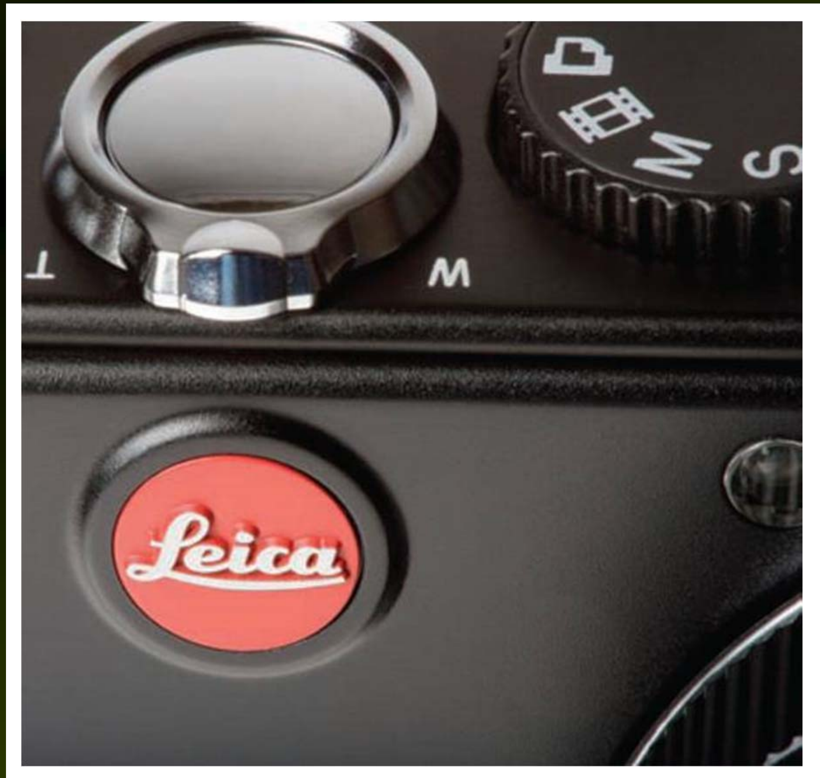
5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS



5.3 Compound Interest

Compound Interest

Compound interest is a natural application of the exponential function to business.

Recall that simple interest is interest that is computed only on the original principal.

Thus, if I denotes the interest on a principal P (in dollars) at an interest rate of r per year for t years, then we have

$$I = Prt$$

The accumulated amount A, the sum of the principal and interest after t years, is given by

$$\begin{aligned} A &= P + I = P + Prt \\ &= P(1 + rt) \qquad \text{Simple interest formula} \end{aligned}$$

Compound Interest

Frequently, interest earned is periodically added to the principal and thereafter earns interest itself at the same rate. This is called compound interest.

Suppose $1000 (the principal) is deposited in a bank for a term of 3 years, earning interest at the rate of 8% per year compounded annually.

Using the simple interest formula we see that the accumulated amount after the first year is

$$\begin{aligned} A_1 &= P(1+rt) \\ &= 1000[1+0.08(1)] \\ &= 1000(1.08) = 1080 \end{aligned}$$

or $1080.

Compound Interest

To find the accumulated amount A_2 at the end of the second year, we use the simple interest formula again, this time with $P = A_1$, obtaining:

$$
\begin{aligned}
A_2 &= P(1+rt) = A_1(1+rt) \\
&= 1000[1+0.08(1)][1+0.08(1)] \\
&= 1000(1+0.08)^2 = 1000(1.08)^2 \approx 1166.40
\end{aligned}
$$

or approximately $1166.40.

Compound Interest

We can use the simple interest formula yet again to find the accumulated amount A_3 at the end of the third year:

$$A_3 = P(1 + rt) = A_2(1 + rt)$$
$$= 1000[1 + 0.08(1)]^2[1 + 0.08(1)]$$
$$= 1000(1 + 0.08)^3 = 1000(1.08)^3 \approx 1259.71$$

or approximately $1259.71.

Compound Interest

Note that the accumulated amounts at the end of each year have the following form:

$$A_1 = 1000(1.08) \qquad A_1 = P(1+r)$$

$$A_2 = 1000(1.08)^2 \qquad \text{or:} \qquad A_2 = P(1+r)^2$$

$$A_3 = 1000(1.08)^3 \qquad A_3 = P(1+r)^3$$

These observations suggest the following general rule:

If *P* dollars are invested over a term of *t* years earning interest at the rate of *r* per year compounded annually, then the accumulated amount is

$$A = P(1+r)^t$$

Compounding More Than Once a Year

The formula

$$A = P(1+r)^t$$

was derived under the assumption that interest was compounded annually.

In practice, however, interest is usually compounded more than once a year.

The interval of time between successive interest calculations is called the conversion period.

Compounding More Than Once a Year

If interest at a nominal a rate of *r* per year is compounded *m* times a year on a principal of *P* dollars, then the simple interest rate per conversion period is

$$i = \frac{r}{m} \qquad \frac{\text{Annual interest rate}}{\text{Periods per year}}$$

For example, the nominal interest rate is 8% per year, and interest is compounded quarterly, then

$$i = \frac{r}{m} = \frac{0.08}{4} = 0.02$$

or 2% per period.

Compounding More Than Once a Year

To find a general formula for the accumulated amount, we apply $A = P(1+r)^t$ repeatedly with the interest rate $i = r/m$.

We see that the accumulated amount at the end of each period is as follows:

$$
\begin{aligned}
&\textbf{First Period:} && A_1 = P(1+i) \\
&\textbf{Second Period:} && A_2 = A_1(1+i) = [P(1+i)](1+i) = P(1+i)^2 \\
&\textbf{Third Period:} && A_3 = A_2(1+i) = [P(1+i)^2](1+i) = P(1+i)^3 \\
&\quad\vdots && \quad\vdots \\
&\boldsymbol{n}\textbf{th Period:} && A_n = A_{n-1}(1+i) = [P(1+i)^{n-1}](1+i) = P(1+i)^n
\end{aligned}
$$

Compound Interest Formula

There are $n = mt$ periods in t years, so the accumulated amount at the end of t year is given by

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

where

A = Accumulated amount at the end of t years

P = Principal

r = Nominal interest rate per year

m = Number of conversion periods per year

t = Term (number of years)

Example 1

Find the accumulated amount after 3 years if $1000 is invested at 8% per year compounded

a. Annually

b. Semiannually

c. Quarterly

d. Monthly

e. Daily

Example 1(a) – *Solution*

Annually:

Here, $P = 1000$, $r = 0.08$, $m = 1$, and $t = 3$, so

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{1}\right)^{(1)(3)}$$

$$= 1000(1.08)^3$$

$$\approx 1259.71$$

or $1259.71.

Example 1(b) – *Solution*

cont'd

Semiannually

Here, $P = 1000$, $r = 0.08$, $m = 2$, and $t = 3$, so

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{2}\right)^{(2)(3)}$$

$$= 1000(1.04)^6$$

$$\approx 1265.32$$

or $1265.32.

Example 1(c) – *Solution*

cont'd

Quarterly:

Here, $P = 1000$, $r = 0.08$, $m = 4$, and $t = 3$, so

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{4}\right)^{(4)(3)}$$

$$= 1000(1.02)^{12}$$

$$\approx 1268.24$$

or \$1268.24.

Example 1(d) – *Solution*

cont'd

Monthly:

Here, $P = 1000$, $r = 0.08$, $m = 12$, and $t = 3$, so

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{12}\right)^{(12)(3)}$$

$$= 1000(1.00667)^{36}$$

$$\approx 1270.24$$

or $1270.24.

Example 1(e) – *Solution*

cont'd

Daily:

Here, P = 1000, r = 0.08, m = 365, and t = 3, so

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{365}\right)^{(365)(3)}$$

$$= 1000(1.00022)^{1095}$$

$$\approx 1271.22$$

or \$1271.22.

Effective Rate of Interest

The last example demonstrates that the interest actually earned on an investment depends on the frequency with which the interest is compounded.

For clarity when comparing interest rates, we can use what is called the *effective rate* (also called the *true rate*):

This is the simple interest rate that would produce the same accumulated amount in 1 year as the nominal rate compounded *m* times a year.

We want to derive a relation between the nominal compounded rate and the effective rate.

Effective Rate of Interest

The accumulated amount after 1 year at a simple interest rate r_{eff} per year is

$$A = P(1 + r_{\text{eff}})$$

The accumulated amount after 1 year at a nominal interest rate r per year compounded m times a year is

$$A = P\left(1 + \frac{r}{m}\right)^m$$

Since $t = 1$

Equating the two expressions gives

$$P(1 + r_{\text{eff}}) = P\left(1 + \frac{r}{m}\right)^m$$

$$1 + r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m$$

Effective Rate of Interest Formula

Solving the last equation for r_{eff} we obtain the formula for computing the effective rate of interest:

$$r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$$

where

r_{eff} = Effective rate of interest

r = Nominal interest rate per year

m = Number of conversion periods per year

Example 2

Find the *effective* rate of interest corresponding to a nominal rate of 8% per year compounded

a. Annually

b. Semiannually

c. Quarterly

d. Monthly

e. Daily

Example 2(a) – *Solution*

Annually:

Let $r = 0.08$ and $m = 1$. Then

$$r_{\text{eff}} = \left(1 + \frac{0.08}{1}\right)^1 - 1$$

$$= 1.08 - 1$$

$$= 0.08$$

or 8%.

Example 2(b) – *Solution*

cont'd

Semiannually:

Let $r = 0.08$ and $m = 2$. Then

$$r_{\text{eff}} = \left(1 + \frac{0.08}{2}\right)^2 - 1$$

$$= 1.0816 - 1$$

$$= 0.0816$$

or 8.16%.

Example 2(c) – *Solution*

cont'd

Quarterly:

Let $r = 0.08$ and $m = 4$. Then

$$r_{\text{eff}} = \left(1 + \frac{0.08}{4}\right)^4 - 1$$

$$= 1.08243 - 1$$

$$\approx 0.08243$$

or 8.243%.

Example 2(d) – *Solution*

cont'd

Monthly:

Let $r = 0.08$ and $m = 12$. Then

$$r_{\text{eff}} = \left(1 + \frac{0.08}{12}\right)^{12} - 1$$

$$= 1.083 - 1$$

$$\approx 0.083$$

or 8.3%.

Example 2(e) – *Solution*

cont'd

Daily:

Let $r = 0.08$ and $m = 365$. Then

$$r_{\text{eff}} = \left(1 + \frac{0.08}{365}\right)^{365} - 1$$

$$= 1.08328 - 1$$

$$\approx 0.08328$$

or 8.328%.

Effective Rate Over Several Years

If the effective rate of interest r_{eff} is known, then the accumulated amount after t years on an investment of P dollars may be more readily computed by using the formula

$$A = P(1 + r_{\text{eff}})^t$$

Present Value

Consider the compound interest formula:

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

The principal *P* is often referred to as the present value, and the accumulated value *A* is called the future value, since it is realized at a future date.

On occasion, an investor may wish to determine how much money he should invest now, at a fixed rate of interest, so that he will realize a certain sum at some future date.

This problem may be solved by expressing *P* in terms of *A*.

Present Value

Present value formula for compound interest

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

Example 3

How much money should be deposited in a bank paying a yearly interest rate of 6% compounded monthly so that after 3 years the accumulated amount will be $20,000?

Solution:

Here, $A = 20{,}000$, $r = 0.06$, $m = 12$, and $t = 3$.

Using the present value formula we get

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

$$= 20{,}000\left(1+\frac{0.06}{12}\right)^{-(12)(3)}$$

$$\approx 16{,}713$$

Example 4

Find the present value of $49,158.60 due in 5 years at an interest rate of 10% per year compounded quarterly.

Solution:

Here, A = 49,158.60, r = 0.1, m = 4, and t = 5.

Using the present value formula we get

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

$$= 49{,}158.6\left(1+\frac{0.1}{4}\right)^{-(4)(5)}$$

$$\approx 30{,}000$$

Continuous Compounding of Interest

One question arises on compound interest:

What happens to the accumulated amount over a fixed period of time if the interest is compounded more and more frequently?

We've seen that the more often interest is compounded, the larger the accumulated amount.

But does the accumulated amount approach a limit when interest is computed more and more frequently?

Continuous Compounding of Interest

Recall that in the compound interest formula

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

the number of conversion periods is *m*.

So, we should let *m* get larger and larger (approach infinity) and see what happens to the accumulated amount *A*.

But first, for clarity, lets rewrite the equation as follows:

$$A = P\left[\left(1+\frac{r}{m}\right)^{m}\right]^{t}$$

Continuous Compounding of Interest

Letting $m \to \infty$, we find that

$$\lim_{m\to\infty} P\left[\left(1+\frac{r}{m}\right)^m\right]^t = P\left[\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m\right]^t$$

We can substitute $u = m/r$ (note that $u \to \infty$ as $m \to \infty$).

Thus

$$P\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^{ur}\right]^t = P\left[\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right]^{rt}$$

Continuous Compounding of Interest

Now, you may recall that

$$\lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^u = e$$

So, we can restate as follows:

$$P\left[\lim_{u \to \infty} \left(1 + \frac{1}{u}\right)^u\right]^{rt} = Pe^{rt}$$

Thus, as the frequency with which interest is compounded increases without bound, the accumulated amount approaches Pe^{rt}.

Continuous Compounding of Interest

Continuous Compound Interest Formula

$$A = Pe^{rt}$$

where

P = Principal

r = Annual interest rate compounded continuously.

t = Time in years.

A = Accumulated amount at the end of t years.

Example 5

Find the accumulated amount after 3 years if $1000 is invested at 8% per year compounded (a) daily, and (b) continuously.

Solution:

a. Using the compound interest formula with $P = 1000$, $r = 0.08$, $m = 365$, and $t = 3$, we find

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

$$= 1000\left(1 + \frac{0.08}{365}\right)^{(365)(3)}$$

$$\approx 1271.22$$

Example 5 – *Solution*

cont'd

b. Using the continuous compound interest formula with $P = 1000$, $r = 0.08$, and $t = 3$, we find

$$A = Pe^{rt} = 1000e^{(0.08)(3)}$$

$$\approx 1271.25$$

Note that both solutions are very similar.

Example 7

How long will it take \$10,000 to grow to \$15,000 if the investment earns an interest rate of 12% per year compounded quarterly?

Solution:

Using the compound interest formula with $A = 15{,}000$, $P = 10{,}000$, $r = 0.12$, and $m = 4$, we obtain

$$15{,}000 = 10{,}000\left(1 + \frac{0.12}{4}\right)^{4t}$$

$$(1.03)^{4t} = \frac{15{,}000}{10{,}000}$$

$$= 1.5$$

Example 7 – *Solution*

cont'd

Taking logarithms on both sides gives

$$\ln(1.03)^{4t} = \ln 1.5$$

$$4t \ln 1.03 = \ln 1.5$$

$$t = \frac{\ln 1.5}{4 \ln 1.03}$$

$$t \approx 3.43$$

So, it will take approximately 3.4 years for the investment to grow from $10,000 to $15,000.

Example 8

Find the interest rate needed for an investment of \$10,000 to grow to an amount of \$18,000 in 5 years if the interest is compounded monthly.

Solution:

Using the compound interest formula with $A = 18{,}000$, $P = 10{,}000$, $m = 12$, and $t = 5$, we find

$$A = P\left(1+\frac{r}{m}\right)^{mt}$$

$$18{,}000 = 10{,}000\left(1+\frac{r}{12}\right)^{(12)(5)}$$

$$\left(1+\frac{r}{12}\right)^{60} = \frac{18{,}000}{10{,}000} = 1.8$$

Example 8 – *Solution*

cont'd

Taking the 60th root on both sides and solving for *r* we get

$$\left(1+\frac{r}{12}\right)^{60}=1.8$$

$$1+\frac{r}{12}=\sqrt[60]{1.8}$$

$$\frac{r}{12}=\sqrt[60]{1.8}-1$$

$$r=12\left(\sqrt[60]{1.8}-1\right)$$

$$\approx 0.009796$$

Example 8 – *Solution*

cont'd

Converting back into an exponential equation,

$$1 + \frac{r}{12} \approx e^{0.009796}$$

$$\approx 1.009844$$

and

$$\frac{r}{12} \approx 1.009844 - 1$$

$$r \approx 0.1181$$

Thus, the interest rate needed is approximately 11.81% per year.