5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.5 Differentiation of Logarithmic Functions

Rule 3: Derivative of the Natural Logarithm

The derivative of ln *x* is

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad (x \neq 0)$$

Example 1(a)

Find the derivative of the function $f(x) = x \ln x$

Solution:

$$f'(x) = x \cdot \frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x)$$

$$= x \cdot \left(\frac{1}{x}\right) + \ln x \cdot (1)$$

$$= 1 + \ln x$$

Example 1(b)

Find the derivative of the function $g(x) = \dfrac{\ln x}{x}$

Solution:

$$g'(x) = \frac{x \cdot \frac{d}{dx}(\ln x) - \ln x \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \cdot \frac{1}{x} - \ln x \cdot (1)}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

Rule 4: Chain Rule for Logarithmic Functions

If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)} \qquad [f(x) > 0]$$

Example 2

Find the derivative of the function $f(x) = \ln(x^2 + 1)$

Solution:

$$f'(x) = \frac{\frac{d}{dx}\left(x^2 + 1\right)}{x^2 + 1}$$

$$= \frac{2x}{x^2 + 1}$$

Example 3

Find the derivative of the function

$$y = \ln[(x^2+1)(x^3+2)^6]$$

Solution:

$$y = \ln[(x^2+1)(x^3+2)^6]$$

$$= \ln(x^2+1) + \ln(x^3+2)^6$$

$$= \ln(x^2+1) + 6\ln(x^3+2)$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(x^2+1)}{x^2+1} + 6\frac{\frac{d}{dx}(x^3+2)}{x^3+2}$$

Example 3 – *Solution*

cont'd

$$= \frac{2x}{x^2+1} + 6\frac{3x^2}{x^3+2}$$

$$= \frac{2x}{x^2+1} + \frac{18x^2}{x^3+2}$$

Logarithmic Differentiation

We have seen how finding derivatives of logarithmic functions becomes easier when applying the laws of logarithms.

These laws can also be used in a process called logarithmic differentiation to permit the differentiation of functions that would be difficult to differentiate or even not be differentiable through other means.

Example 5

Use logarithmic differentiation to find the derivative of

$$y = x(x+1)(x^2+1)$$

Solution:

Take the natural logarithm of both sides of the equation:

$$\ln y = \ln[x(x+1)(x^2+1)]$$

Use the laws of logarithms to rewrite the equation:

$$\ln y = \ln(x) + \ln(x+1) + \ln(x^2+1)$$

Differentiate both sides of the equation:

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln(x) + \frac{d}{dx}\ln(x+1) + \frac{d}{dx}\ln(x^2+1)$$

Example 5 – *Solution*

cont'd

$$= \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$$

On the left side, note that *y* is a function of *x*, therefore:

$$y = f(x)$$

$$\ln y = \ln[f(x)]$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln[f(x)]$$

$$= \frac{f'(x)}{f(x)}$$

$$= \frac{y'}{y}$$

Example 5 – *Solution*

cont'd

Thus, we have:

$$\frac{d}{dx}\ln y = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$$

$$\frac{y'}{y} = \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$$

Finally, solving for *y*′ we get:

$$y' = y\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)$$

$$= x(x+1)(x^2+1)\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)$$

Logarithmic Differentiation

To find *dy/dx* by logarithmic differentiation:

1. Take the natural logarithm on both sides of the equation and use the properties of logarithms to write any “complicated expression” as a sum of simpler terms.
2. Differentiate both sides of the equation with respect to *x*.
3. Solve the resulting equation for *dy/dx*.

Example 6

Use logarithmic differentiation to find the derivative of

$$y = x^2(x-1)(x^2+4)^3$$

Solution:

Take the natural logarithm of both sides of the equation and use the laws of logarithms to rewrite the equation:

$$\ln y = \ln[x^2(x-1)(x^2+4)^3]$$

$$= \ln(x^2) + \ln(x-1) + \ln(x^2+4)^3$$

$$= 2\ln x + \ln(x-1) + 3\ln(x^2+4)$$

Example 6 – *Solution*

cont'd

Differentiate both sides of the equation:

$$\frac{d}{dx}\ln y = 2\frac{d}{dx}\ln x + \frac{d}{dx}\ln(x-1) + 3\frac{d}{dx}\ln(x^2+4)$$

$$= 2\cdot\frac{1}{x} + \frac{1}{x-1} + 3\cdot\frac{2x}{x^2+4}$$

$$= \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}$$

Example 6 – *Solution*

cont'd

Solve for *dy/dx*:

$$\frac{d}{dx}\ln y = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}$$

$$\frac{y'}{y} = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}$$

$$y' = y\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$$

$$= x^2(x-1)(x^2+4)^3\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+4}\right)$$

Example 7

Use logarithmic differentiation to find the derivative of

$$f(x) = x^x \qquad (x > 0)$$

Solution:

Take the natural logarithm of both sides of the equation and use the laws of logarithms to rewrite the equation:

$$\ln f(x) = \ln x^x$$
$$= x \ln x$$

Example 7 – *Solution*

cont'd

Differentiate both sides of the equation:

$$\frac{d}{dx}\ln f(x) = x \cdot \frac{d}{dx}(\ln x) + \ln x \cdot \frac{d}{dx}(x)$$

$$= x \cdot \frac{1}{x} + \ln x \cdot (1)$$

$$= 1 + \ln x$$

Example 7 – *Solution*

cont'd

Solve for *dy/dx*:

$$\frac{d}{dx}\ln f(x) = 1 + \ln x$$

$$\frac{f'(x)}{f(x)} = 1 + \ln x$$

$$f'(x) = f(x)(1 + \ln x)$$

$$= x^x(1 + \ln x)$$