5 EXPONENTIAL AND LOGARITHMIC FUNCTIONS

5.6

Exponential Functions as Mathematical Models

Applied Example 1 – *Growth of Bacteria*

Under a laboratory, the number of bacteria in a culture grows according to

$$Q(t) = Q_0 e^{kt}$$

where Q_0 denotes the number of bacteria initially present in the culture, k is a constant determined by the strain of bacteria under consideration, and t is the elapsed time measured in hours.

Applied Example 1 – *Growth of Bacteria* cont'd

Suppose 10,000 bacteria are present initially in the culture and 60,000 present two hours later.

a. How many bacteria will there be in the culture at the end of four hours?

a. What is the rate of growth of the population after four hours?

Applied Example 1(a) – *Solution*

We are given that $Q(0) = Q_0 = 10{,}000$, so $Q(t) = 10{,}000e^{kt}$.
At $t = 2$ there are 60,000 bacteria, so $Q(2) = 60{,}000$, thus:

$$Q(t) = Q_0 e^{kt}$$

$$60{,}000 = 10{,}000 e^{2k}$$

$$e^{2k} = 6$$

Taking the natural logarithm on both sides we get:

$$\ln e^{2k} = \ln 6$$

$$2k = \ln 6$$

$$k \approx 0.8959$$

Applied Example 1(a) – *Solution* cont'd

So, the number of bacteria present at any time *t* is given by:

$$Q(t) = 10,000e^{0.8959t}$$

At the end of four hours ($t = 4$), there will be

$$Q(4) = 10,000e^{0.8959(4)}$$

$$\approx 360,029$$

or 360,029 bacteria.

Applied Example 1(b) – *Solution*

cont'd

The rate of growth of the bacteria at any time t is given by

$$Q'(t) = kQ(t)$$

Using the result from part (a), we find that the rate of bacterial growth at the end of four hours is

$$Q'(4) = kQ(4)$$

$$\approx (0.8959)(360,029)$$

$$\approx 322,550$$

or approximately 322,550 bacteria per hour.

Applied Example 2 – *Radioactive Decay*

Radioactive substances decay exponentially. For example, the amount of radium present at any time t obeys the law

$$Q(t) = Q_0 e^{-kt} \qquad (0 \le t < \infty)$$

where Q_0 is the initial amount present and k is a suitable positive constant. The half-life of a radioactive substance is the time required for a given amount to be reduced by one-half. The half-life of radium is approximately 1600 years.

Suppose initially there are 200 milligrams of pure radium.

a. Find the amount left after t years.

b. What is the amount after 800 years?

Applied Example 2(a) – *Solution*

The initial amount is 200 milligrams, so $Q(0) = Q_0 = 200$, so

$$Q(t) = 200e^{-kt}$$

The half-life of radium is 1600 years, so $Q(1600) = 100$, thus

$$100 = 200e^{-1600k}$$

$$e^{-1600k} = \frac{1}{2}$$

Applied Example 2(a) – *Solution* cont'd

Taking the natural logarithm on both sides yields:

$$\ln e^{-1600k} = \ln \frac{1}{2}$$

$$-1600k \ln e = \ln \frac{1}{2}$$

$$-1600k = \ln \frac{1}{2}$$

$$k = -\frac{1}{1600} \ln \frac{1}{2} \approx 0.0004332$$

Therefore, the amount of radium left after t years is:

$$Q(t) = 200e^{-0.0004332t}$$

Applied Example 2(b) – *Solution*

cont'd

In particular, the amount of radium left after 800 years is:

$$Q(800) = 200e^{-0.0004332(800)}$$

$$\approx 141.42$$

or approximately 141 milligrams.

Applied Example 5 – *Assembly Time*

The Camera Division of Eastman Optical produces a single lens reflex camera. Eastman's training department determines that after completing the basic training program, a new, previously inexperienced employee will be able to assemble

$$Q(t) = 50 - 30e^{-0.5t}$$

model F cameras per day, t months after the employee starts work on the assembly line.

Applied Example 5 – *Assembly Time* cont'd

a. How many model F cameras can a new employee assemble per day after basic training?

b. How many model F cameras can an employee with one month of experience assemble per day?

c. How many model F cameras can the average experienced employee assemble per day?

Applied Example 5 – *Solution*

a. The number of model F cameras a new employee can assemble is given by

$$Q(0) = 50 - 30 = 20$$

b. The number of model F cameras that an employee with 1, 2, and 6 months of experience can assemble per day is given by

$$Q(1) = 50 - 30e^{-0.5(1)} \approx 31.80$$

or about 32 cameras per day.

Applied Example 5 – *Solution*

cont'd

c. As t increases without bound, $Q(t)$ approaches 50. Hence, the average experienced employee can be expected to assemble 50 model F cameras per day.