6 INTEGRATION

6.1 Antiderivatives and the Rules of Integration

Antiderivatives

Recall the Maglev problem discussed in chapter 2.

The question asked then was:

- If we know the position of the maglev at any time t, can we find its velocity at that time?

- The position was described by $f(t)$, and the velocity by $f'(t)$.

Antiderivatives

Now, in Chapters 6 and 7 we will consider precisely the opposite problem:

- If we know the velocity of the maglev at any time t, can we find its position at that time?

- That is, knowing its velocity function $f'(t)$, can we find its position function $f(t)$?

Antiderivatives

To solve this kind of problems, we need the concept of the antiderivative of a function.

A function F is an antiderivative of f on an interval I if $F'(t) = f(t)$ for all of t in I.

Example 1

Let $F(x) = \frac{1}{3}x^3 - 2x^2 + x - 1$. Show that *F* is an antiderivative of $f(x) = x^2 - 4x + 1$

Solution:

Differentiating the function *F*, we obtain

$$F'(x) = x^2 - 4x + 1 = f(x)$$

and the desired result follows.

Example 2

Let $F(x) = x$, $G(x) = x + 2$, $H(x) = x + C$, where C is a constant. Show that F, G, and H are all antiderivatives of the function f defined by $f(x) = 1$.

Solution:

Since

$$F'(x) = \frac{d}{dx}(x) = 1 = f(x)$$

$$G'(x) = \frac{d}{dx}(x+2) = 1 = f(x)$$

$$H'(x) = \frac{d}{dx}(x+C) = 1 = f(x)$$

we see that F, G, and H are indeed antiderivatives of f.

Theorem 1

Let G be an antiderivative of a function f.

Then, every antiderivative F of f must be of the form

$$F(x) = G(x) + C$$

where C is a constant.

Example 3

Prove that the function $G(x) = x^2$ is an antiderivative of the function $f(x) = 2x$. Write a general expression for the antiderivatives of f.

Solution:

Since $G'(x) = 2x = f(x)$, we have shown that $G(x) = x^2$ is an antiderivative of $f(x) = 2x$.

By Theorem 1, every antiderivative of the function $f(x) = 2x$ has the form $F(x) = x^2 + C$, where C is a constant.

The Indefinite Integral

The process of finding all the antiderivatives of a function is called antidifferentiation or integration.

We use the symbol $\int$, called an integral sign, to indicate that the operation of integration is to be performed on some function f.

Thus,

$$\int 1\,dx = x + C \quad \textbf{and} \quad \int 2x\,dx = x^2 + K$$

where C and K are arbitrary constants.

Basic Integration Rules

Rule 1: The Indefinite Integral of a Constant

$$\int k\,dx = kx + C \qquad (k, \text{a constant})$$

Example 4

Find each of the following indefinite integrals:

a. $\int 2dx$ b. $\int \pi^2 dx$

Solution:

Each of the integrals had the form $f(x) = k$, where k is a constant.

Applying Rule 1 in each case yields:

a. $\int 2dx = 2x + C$

b. $\int \pi^2 dx = \pi^2 x + C$

Basic Integration Rules

From the rule of differentiation,

$$\frac{d}{dx}x^n = nx^{n-1}$$

we obtain the following rule of integration:

Rule 2: The Power Rule

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad (n \neq -1)$$

Example 5(a)

Find the indefinite integral: $\int x^3 dx$

Solution:

$$\int x^3 dx = \frac{1}{4}x^4 + C$$

Example 5(b)

Find the indefinite integral: $\int x^{3/2} dx$

Solution:

$$\int x^{3/2} dx = \frac{1}{\frac{5}{2}} x^{5/2} + C$$

$$= \frac{2}{5} x^{5/2} + C$$

Example 5(c)

Find the indefinite integral: $\int \frac{1}{x^{3/2}} dx$

Solution:

$$\int \frac{1}{x^{3/2}} dx = \int x^{-3/2} dx$$

$$= \frac{1}{-\frac{1}{2}} x^{-1/2} + C$$

$$= -2x^{-1/2} + C$$

$$= -\frac{2}{x^{1/2}} + C$$

Basic Integration Rules

Rule 3: The Indefinite Integral of a Constant Multiple of a Function

$$\int cf(x)dx = c\int f(x)dx$$

where c is a constant.

Example 6(a)

Find the indefinite integral: $\int 2t^3 dt$

Solution:

$$\int 2t^3 dt = 2\int t^3 dt$$

$$= 2\left(\frac{1}{4}t^4 + K\right)$$

$$= \frac{1}{2}t^4 + 2K$$

$$= \frac{1}{2}t^4 + C$$

Example 6(b)

Find the indefinite integral: $\int -3x^{-2}dx$

Solution:

$$\int -3x^{-2}dx = -3\int x^{-2}dx$$

$$= -3(-1)x^{-1} + C$$

$$= \frac{3}{x} + C$$

Basic Integration Rules

Rule 4: The Sum Rule

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

Example 7

Find the indefinite integral: $\int \left(3x^5 + 4x^{3/2} - 2x^{-1/2}\right) dx$

Solution:

$$\int \left(3x^5 + 4x^{3/2} - 2x^{-1/2}\right) dx = \int 3x^5 dx + \int 4x^{3/2} dx - \int 2x^{-1/2} dx$$

$$= 3\int x^5 dx + 4\int x^{3/2} dx - 2\int x^{-1/2} dx$$

$$= 3\left(\frac{1}{6}\right) x^6 + 4\left(\frac{2}{5}\right) x^{5/2} - 2(2) x^{1/2} + C$$

$$= \frac{1}{2} x^6 + \frac{8}{5} x^{5/2} - 4x^{1/2} + C$$

Basic Integration Rules

Rule 5: The Indefinite Integral of the Exponential Function

$$\int e^x dx = e^x + C$$

Example 8

Find the indefinite integral: $\int (2e^x - x^3)dx$

Solution:

$$\int (2e^x - x^3)dx = \int 2e^x dx - \int x^3 dx$$

$$= 2\int e^x dx - \int x^3 dx$$

$$= 2e^x - \frac{1}{4}x^4 + C$$

Basic Integration Rules

Rule 6: The Indefinite Integral of the Function $f(x) = x^{-1}$

$$\int x^{-1} dx = \int \frac{1}{x} dx = \ln|x| + C \qquad (x \neq 0)$$

Example 9

Find the indefinite integral: $\int \left(2x + \frac{3}{x} + \frac{4}{x^2} \right) dx$

Solution:

$$\int \left(2x + \frac{3}{x} + \frac{4}{x^2} \right) dx = \int 2x\,dx + \int \frac{3}{x} dx + \int \frac{4}{x^2} dx$$

$$= 2\int x\,dx + 3\int \frac{1}{x} dx + 4\int x^{-2} dx$$

$$= 2\left(\frac{1}{2} \right) x^2 + 3\ln|x| + 4(-1)x^{-1} + C$$

$$= x^2 + 3\ln|x| - \frac{4}{x} + C$$

Differential Equations

Given the derivative of a function, f', can we find the function f? Consider the function $f'(x) = 2x – 1$ from which we want to find $f(x)$.

We can find f by integrating the equation:

$$\int f'(x)dx = \int (2x-1)dx = x^2 - x + C$$

where C is an arbitrary constant.

Thus, infinitely many functions have the derivative f', each differing from the other by a constant.

Differential Equations

Equation $f'(x) = 2x - 1$ is called a differential equation.

In general, a differential equation involves the derivative of an unknown function.

A solution of a differential equation is any function that satisfies the differential equation.

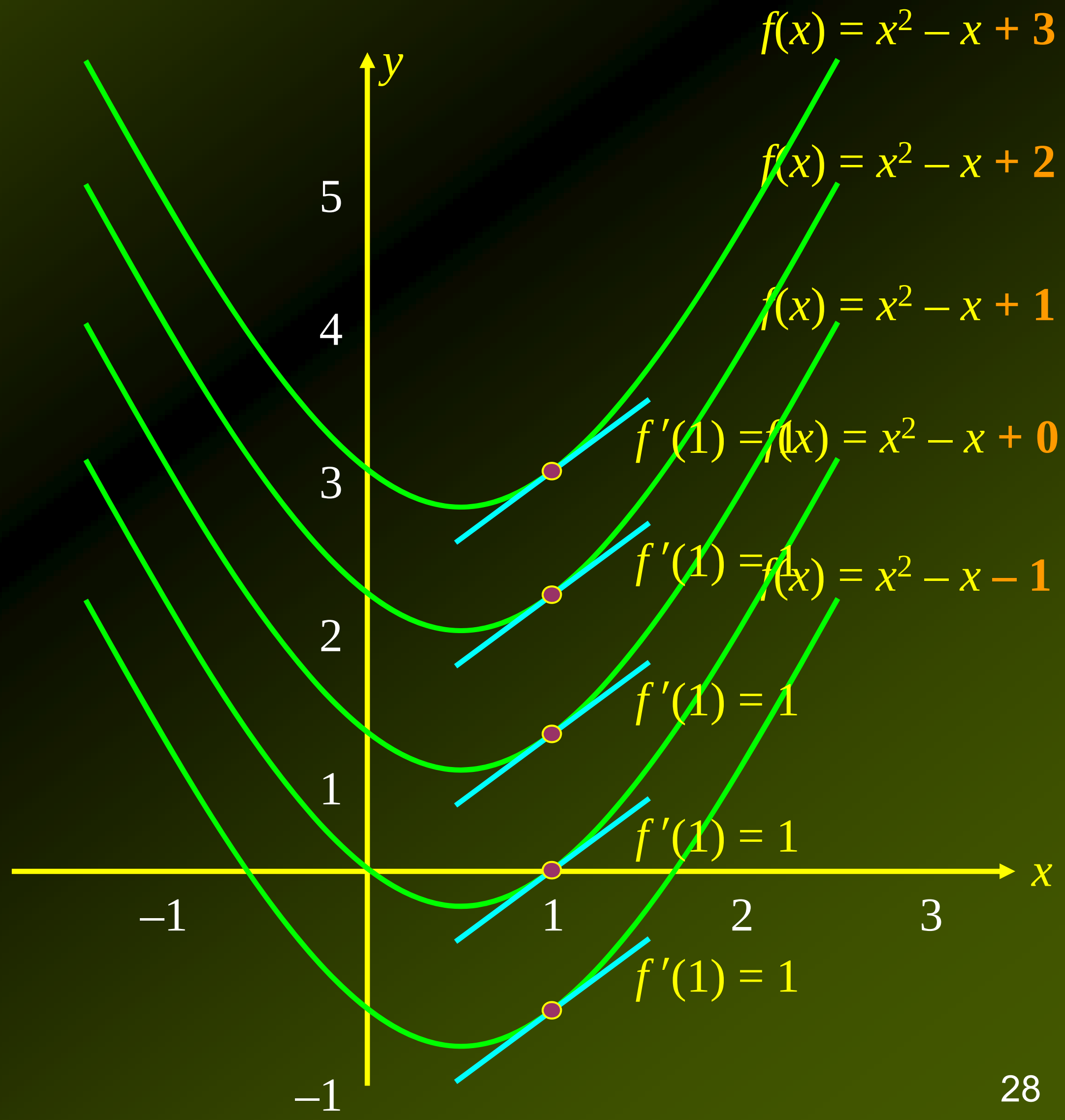
For the case of $f'(x) = 2x - 1$, we find that $f(x) = x^2 - x + C$ gives *all* the solutions of the differential equation, and it is therefore called the general solution of the differential equation.

Differential Equations

Different values of C yield different functions $f(x)$.

But all these functions have the same slope for any given value of x.

For example, for any value of C, we always find that $f'(1) = 1$.



Differential Equations

It is possible to obtain a particular solution by specifying the value the function must assume for a given value of x.

For example, suppose we know the function f must pass through the point (1, 2), which means $f(1) = 2$.

Using this condition on the general solution we can find the value of C:

$$f(1) = 1^2 - 1 + C = 2$$

$$C = 2$$

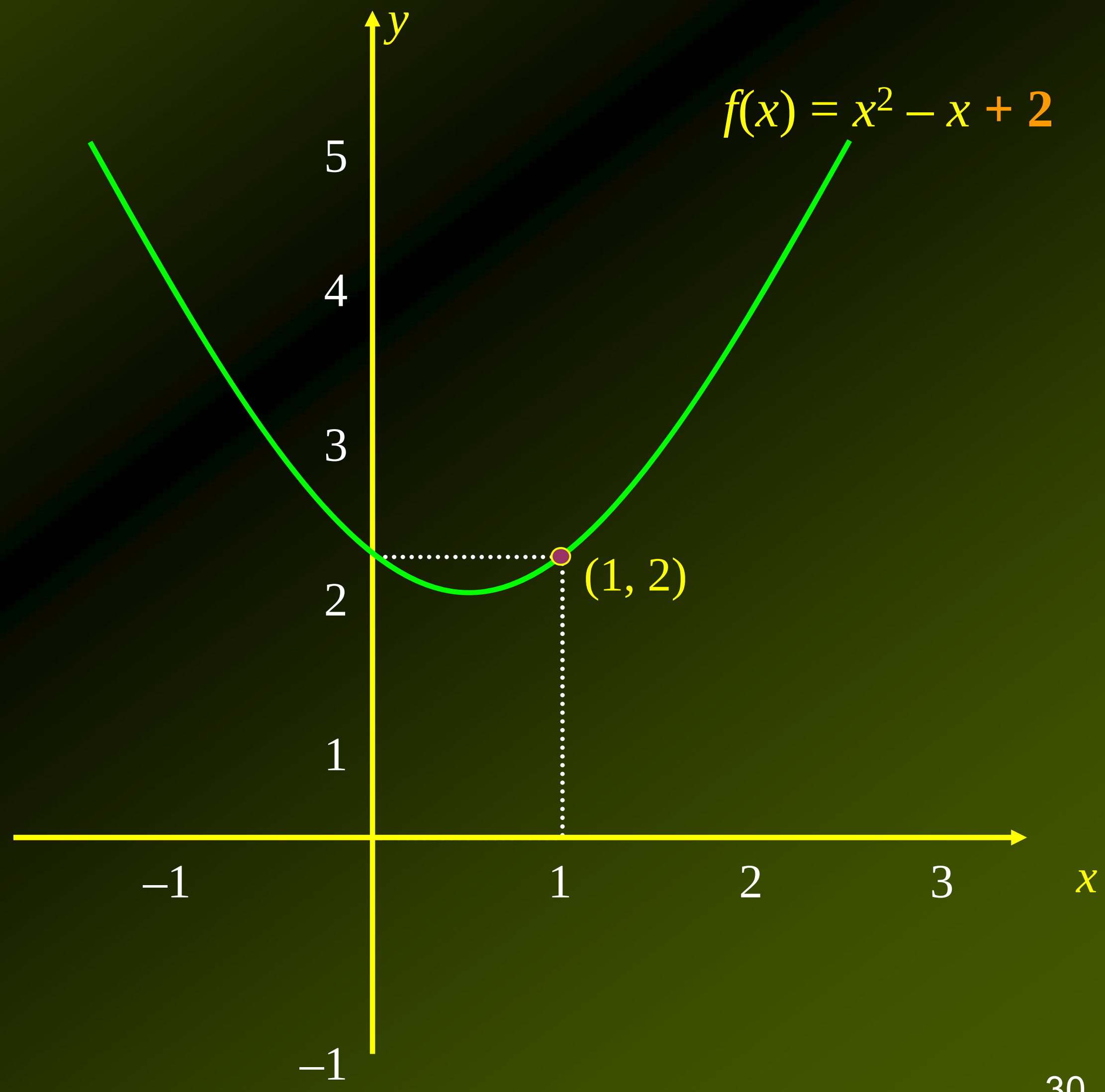
Thus, the particular solution is

$$f(x) = x^2 - x + 2$$

Differential Equations

Here is the graph of the particular solution of f when $C = 2$.

Note that this graph does go through the point (1, 2).



Initial Value Problems

The problem we just discussed is of a type called initial value problem.

In this type of problem we are required to find a function satisfying

1. A differential equation.
2. One or more initial conditions.

Example 10

Find the function *f* if it is known that

$$f'(x) = 3x^2 - 4x + 8 \quad \text{and} \quad f(1) = 9$$

Solution:

Integrating the function f', we find

$$f(x) = \int f'(x)dx$$

$$= \int (3x^2 - 4x + 8)dx$$

$$= x^3 - 2x^2 + 8x + C$$

Example 10 – *Solution*

cont'd

Using the condition $f(1) = 9$, we have

$$f(x) = x^3 - 2x^2 + 8x + C$$

$$9 = f(1) = (1)^3 - 2(1)^2 + 8(1) + C$$

$$9 = 7 + C$$

$$C = 2$$

Therefore, the required function f is

$$f(x) = x^3 - 2x^2 + 8x + 2$$

Applied Example 11 – *Velocity of Maglev*

In a test run of a maglev, data obtained from reading its speedometer indicate that the velocity of the maglev at time t can be described by the velocity function

$$v(t) = 8t \qquad (0 \leq t \leq 30)$$

Find the position function of the maglev. Assume that initially the maglev is located at the origin of a coordinate line.

Applied Example 11 – *Solution*

Let $s(t)$ denote the position of the maglev at any given time t $(0 \le t \le 30)$. Then, $s'(t) = v(t)$.

So, we have the initial value problem

$$\left.\begin{aligned} s'(t) &= 8t \\ s(0) &= 0 \end{aligned}\right\}$$

Integrating the function s', we find

$$\begin{aligned} s(t) &= \int s'(t)\,dt \\ &= \int 8t\,dt \\ &= 4t^2 + C \end{aligned}$$

Applied Example 11 – *Solution*

cont'd

Using the condition $s(0) = 0$, we have

$$s(t) = 4t^2 + C$$

$$0 = s(0) = 4(0) + C$$

$$0 = 0 + C$$

$$C = 0$$

Therefore, the required function s is

$$s(t) = 4t^2 \qquad (0 \le t \le 30)$$