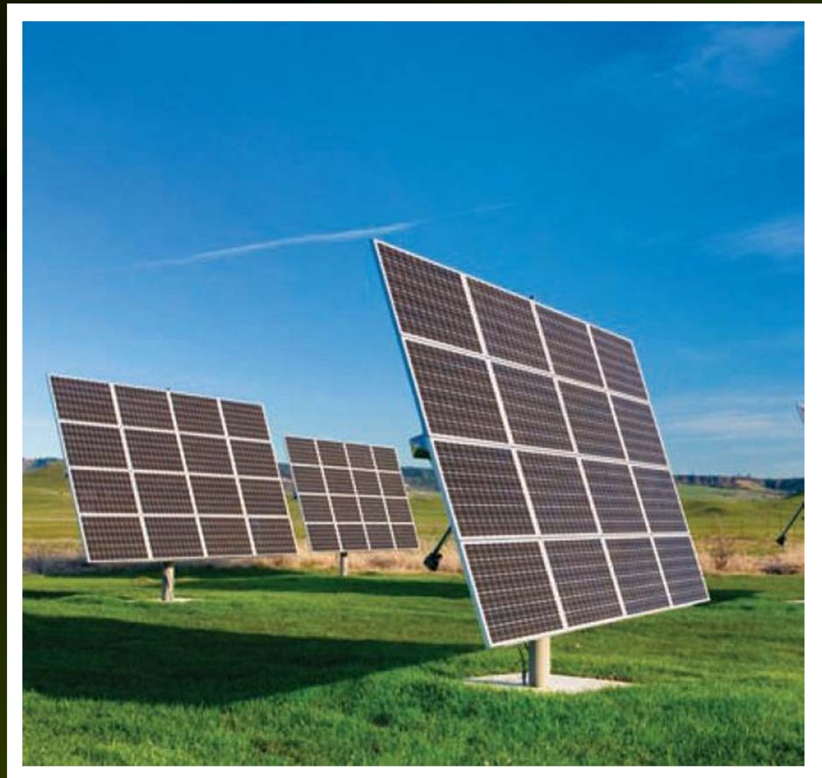
6 INTEGRATION



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6.2 Integration by Substitution

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Integration by Substitution

The method of substitution is related to the chain rule for differentiating functions.

It is a powerful tool for integrating a large class of functions.

How the Method of Substitution Works

Consider the indefinite integral

$$\int 2(2x+4)^5 dx$$

One way to solve this integral is to expand the expression and integrate the resulting integrand term by term.

An alternative approach simplifies the integral by making a change of variable.

Write $u = 2x + 4$

with differential $du = 2dx$

How the Method of Substitution Works

Substitute *u* = 2*x* + 4 and *du* = 2*dx* in the original expression:

$$\int 2(2x+4)^5 dx = \int (2x+4)^5 (2dx) = \int u^5 du$$

Now it's easy to integrate:

$$\int u^5 du = \frac{1}{6}u^6 + C$$

Replacing *u* by *u* = 2*x* + 4, we obtain:

$$\int 2(2x+4)^5 dx = \frac{1}{6}u^6 + C$$

$$= \frac{1}{6}(2x+4)^6 + C$$

How the Method of Substitution Works

We can verify the result by finding its derivative:

$$\frac{d}{dx}\left[\frac{1}{6}(2x+4)^6 + C\right] = \frac{1}{6}\cdot 6 \cdot (2x+4)^5 \cdot (2)$$

$$= 2(2x+4)^5$$

The derivative is indeed the original integrand expression.

The Method of Integration by Substitution

Step 1 Let $u = g(x)$, where $g(x)$ is part of the integrand, usually the "inside function" of the composite function $f(g(x))$.

Step 2 Find $du = g'(x)dx$.

Step 3 Use the substitution $u = g(x)$ and $du = g'(x)dx$ to convert the entire integral into one involving only u.

Step 4 Evaluate the resulting integrand.

Step 5 Replace u by $g(x)$ to obtain the final solution as a function of x.

Example 1

Find $\int 2x(x^2+3)^4 dx$

Solution:

Step 1 The integrand involves the composite function

$$(x^2+3)^4$$

with “inside function”

$$g(x) = x^2 + 3$$

So, we choose

$$u = x^2 + 3$$

Example 1 – *Solution*

cont'd

Step 2 Find *du/dx* and solve for *du*:

$$u = x^2 + 3$$

$$\frac{du}{dx} = 2x$$

$$du = 2xdx$$

Step 3 Substitute $u = x^2 + 3$ and $du = 2xdx$, to obtain an integral involving only *u*:

$$\int 2x(x^2+3)^4\,dx = \int (x^2+3)^4(2xdx)$$

$$= \int u^4\,du$$

Example 1 – *Solution*

cont'd

Step 4 Evaluate the integral:

$$\int u^4 du = \frac{1}{5}u^5 + C$$

Step 5 Replace u by $x^2 + 3$ to find the solution:

$$\int 2x(x^2+3)^4 dx = \frac{1}{5}(x^2+3)^5 + C$$

Example 4

Find $\int e^{-3x}dx$

Solution:

Let $u = -3x$, so that $du = -3dx$, or $dx = -\frac{1}{3}du$.

Substitute to express the integrand in terms of u:

$$\int e^{-3x}dx = \int e^{u}\left(-\frac{1}{3}du\right)$$

$$= -\frac{1}{3}\int e^{u}du$$

Example 4 – *Solution*

cont'd

Evaluate the integral:

$$-\frac{1}{3}\int e^u du = -\frac{1}{3}e^u + C$$

Replace *u* by –3*x* to find the solution:

$$\int e^{-3x}dx = -\frac{1}{3}e^{-3x} + C$$

Example 5

Find $\int \frac{x}{3x^2+1}dx$

Solution:

Let $u = 3x^2 + 1$, so that $du = 6xdx$, or $xdx = \frac{1}{6}du$.

Substitute to express the integrand in terms of u:

$$\int \frac{x}{3x^2+1}dx = \int \frac{1}{3x^2+1}(xdx)$$

$$= \int \frac{1}{u}\cdot\frac{1}{6}du$$

$$= \frac{1}{6}\int \frac{1}{u}\, du$$

Example 5 – *Solution*

cont'd

Evaluate the integral:

$$\frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + C$$

Replace *u* by $3x^2 + 1$ to find the solution:

$$\int \frac{x}{3x^2+1}\,dx = \frac{1}{6}\ln|3x^2+1| + C$$

Example 6

Find $\int \frac{(\ln x)^2}{2x} dx$

Solution:

Let $u = \ln x$, so that $du = 1/x\, dx$, or $dx/x = du$.

Substitute to express the integrand in terms of u:

$$\int \frac{(\ln x)^2}{2x} dx = \frac{1}{2}\int (\ln x)^2 (\frac{dx}{x})$$

$$= \frac{1}{2}\int u^2 du$$

Example 6 – *Solution*

cont'd

Evaluate the integral:

$$\frac{1}{2}\int u^2 du = \frac{1}{2} \cdot \frac{1}{3} u^3 + C$$

$$= \frac{1}{6} u^3 + C$$

Replace u by $\ln x$ to find the solution:

$$\int \frac{(\ln x)^2}{2x} dx = \frac{1}{6} (\ln x)^3 + C$$