6 INTEGRATION

6.4 The Fundamental Theorem of Calculus

Theorem 2: The Fundamental Theorem of Calculus

Let *f* be continuous on [*a*, *b*]. Then,

$$\int_a^b f(x)dx = F(b) - F(a)$$

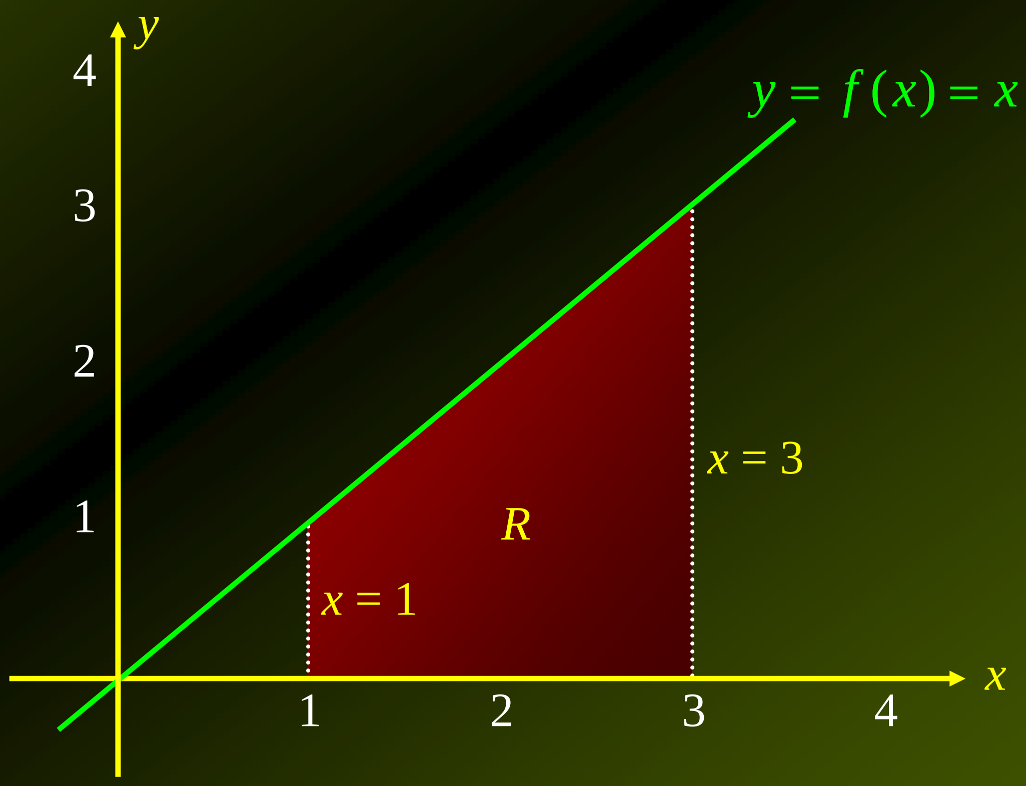
where *F* is any antiderivative of *f*; that is, $F'(x) = f(x)$.

Example 1

Let R be the region under the graph of $f(x) = x$ on the interval $[1, 3]$. Use the fundamental theorem of calculus to find the area A of R and verify your result by elementary means.

Example 1 – *Solution*

The graph shows the region to be evaluated. Since f is nonnegative on [1, 3], the area of R is given by the definite integral of f from 1 to 3.



Example 1 – *Solution*

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By the fundamental theorem of calculus, we have

$$A = \int_1^3 x\,dx = \frac{1}{2}x^2 + C\Big|_1^3$$

$$= \left(\frac{9}{2} + C\right) - \left(\frac{1}{2} + C\right)$$

$$= 4$$

Thus, the area *A* of region *R* is 4 square units.

Note that the constant of integration *C* dropped out.

This is true in general.

Example 1 – *Solution*

cont'd

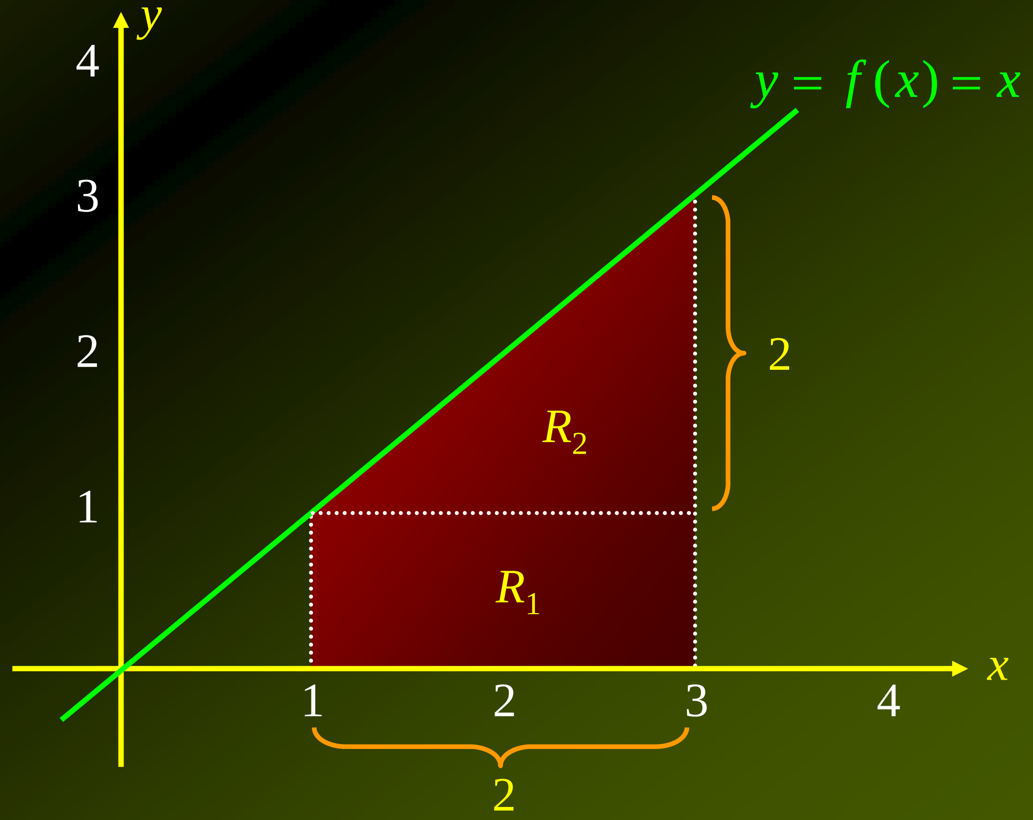
Using elementary means, note that area A is equal to R_1 (base × height) plus R_2 ($\frac{1}{2}$ × base × height).

Thus,

$A = R_1 + R_2$

$= 2(1) + \frac{1}{2}(2)(2)$

$= 4$

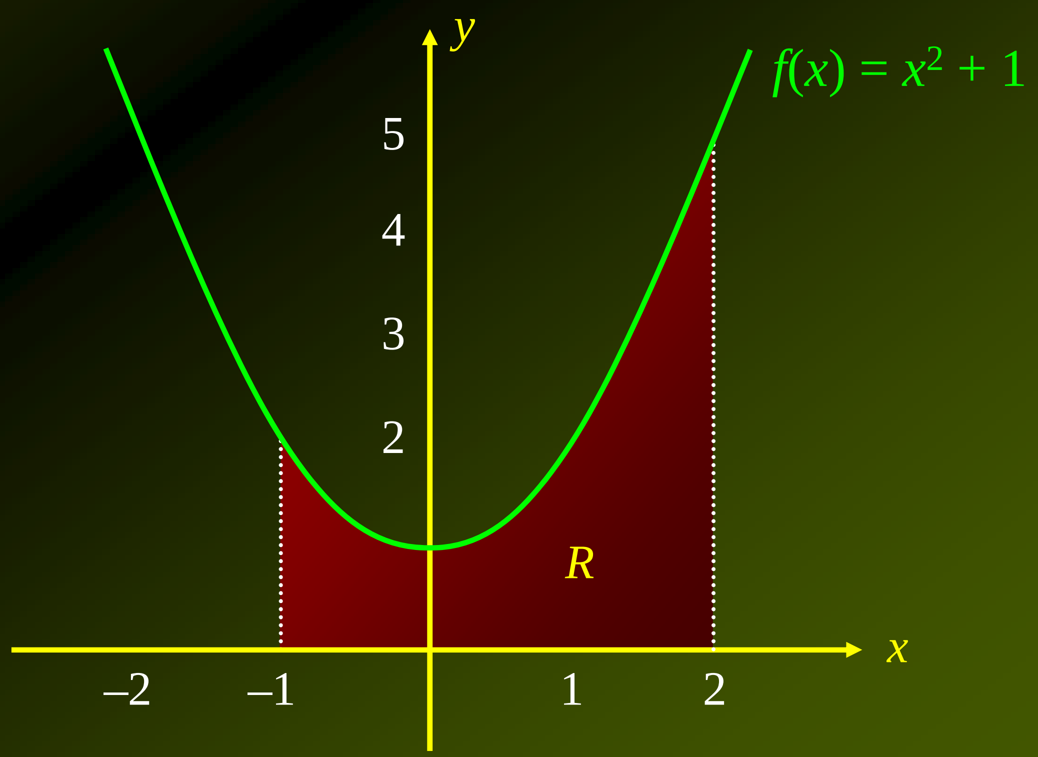


Example 3

Find the area of the region under the graph of $y = x^2 + 1$ from $x = –1$ to $x = 2$.

Solution:

Note below that the full region R under consideration lies above the x axis.



Example 3 – *Solution*

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Using the fundamental theorem of calculus, we find that the required area is

$$\int_{-1}^{2}(x^2+1)\,dx = \left(\frac{1}{3}x^3 + x\right)\Bigg|_{-1}^{2}$$

$$= \left[\frac{1}{3}(2)^3 + (2)\right] - \left[\frac{1}{3}(-1)^3 + (-1)\right]$$

$$= \frac{8}{3} + 2 + \frac{1}{3} + 1$$

$$= 6$$

or 6 square units.

Net Change Formula

The net change in a function *f* over an interval [*a*, *b*] is given by

$$f(b) - f(a) = \int_a^b f'(x)\,dx$$

provided *f′* is continuous on [*a*, *b*].

Applied Example 6 – *Population Growth in Clark County*

Clark County, Nevada, (dominated by Las Vegas) is the fastest growing metropolitan area in the United States. From 1970 through 2000, the population was growing at a rate of

$$R(t) = 133{,}680t^2 - 178{,}788t + 234{,}633 \qquad (0 \le t \le 3)$$

people per decade, where $t = 0$ corresponds to the beginning of 1970.

What was the net change in population over the decade from 1980 to 1990?

Applied Example 6 – *Solution*

The net change in population over the decade from 1980 to 1990 is given by $P(2) - P(1)$, where P denotes the population in the county at time t. But $P' = R$, and so the net change in population is

$$P(2) - P(1) = \int_1^2 P'(t)dt = \int_1^2 R(t)dt$$

$$= \int_1^2 (133,680t^2 - 178,788t + 234,633)dt$$

$$= 44,560t^3 - 89,394t^2 + 234,633t \Big|_1^2$$

$$= [44,560(2)^3 - 89,394(2)^2 + 234,633(2)]$$

$$-[44,560(1)^3 - 89,394(1)^2 + 23,4633(1)]$$

$$= 278,371$$

Applied Example 8 – *Assembly Time of Workers*

An efficiency study conducted for Elektra Electronics showed that the rate at which Space Commander walkie-talkies are assembled by the average worker *t* hours after starting work at 8:00 a.m. is given by the function

$$f(t) = -3t^2 + 12t + 15 \qquad (0 \le t \le 4)$$

Determine how many walkie-talkies can be assembled by the average worker in the first hour of the morning shift.

Applied Example 8 – *Solution*

Let $N(t)$ denote the number of walkie-talkies assembled by the average worker t hours after starting work in the morning shift. Then, we have

$$N'(t) = f(t) = -3t^2 + 12t + 15$$

Therefore, the number of units assembled by the average worker in the first hour of the morning shift is

$$\begin{aligned} N(1) - N(0) &= \int_0^1 N'(t)\,dt = \int_0^1 (-3t^2 + 12t + 15)\,dt \\ &= (-t^3 + 6t^2 + 15t)\Big|_0^1 = -1 + 6 + 15 \\ &= 20 \end{aligned}$$

or 20 units.