6

INTEGRATION

6.5 Evaluating Definite Integrals

Properties of the Definite Integral

Let *f* and *g* be integrable functions, then

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx \qquad (c,\ a\ \ \text{constant})$

4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (a < c < b)$

Example 1 – *Using the Method of Substitution*

Evaluate $\int_0^4 x\sqrt{9+x^2}\,dx$

Solution:

First, find the indefinite integral: $I = \int x\sqrt{9+x^2}\,dx$

1. Let $u = 9 + x^2$ so that

$$du = \frac{d}{dx}(9+x^2)dx$$

$$= 2xdx$$

$$xdx = \frac{1}{2}du$$

Example 1 – *Solution*

cont'd

2. Then, integrate by substitution using $x dx = \frac{1}{2} du$:

$$I = \int x\sqrt{9 + x^2}\, dx$$

$$= \int \frac{1}{2}\sqrt{u}\, du$$

$$= \frac{1}{2}\int u^{1/2}\, du$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(9 + x^2)^{3/2} + C$$

Example 1 – *Solution*

cont'd

Using the results, we evaluate the definite integral:

$$\int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{3}(9+x^2)^{3/2}\Big|_0^4$$

$$= \frac{1}{3}[(9+(4)^2)^{3/2} - (9+(0)^2)^{3/2}]$$

$$= \frac{1}{3}(125-27)$$

$$= \frac{98}{3}$$

$$= 32\tfrac{2}{3}$$

Example 3 – *Using the Method of Substitution*

Evaluate $\int_0^1 \frac{x^2}{x^3+1}\,dx$

Solution:

Let $u = x^3 + 1$ so that

$$du = \frac{d}{dx}(x^3+1)dx$$

$$= 3x^2 dx$$

$$\frac{1}{3}du = x^2 dx$$

Example 3 – *Solution*

cont'd

Find the lower and upper limits of integration with respect to u:

- When $x = 0$, the lower limit is $u = (0)^3 + 1 = 1$.
- When $x = 1$, the upper limit is $u = (1)^3 + 1 = 2$.

Substitute $x^2dx = \frac{1}{3}du$, along with the limits of integration:

$$\int_0^1 \frac{x^2}{x^3+1}dx = \int_0^1 \frac{1}{x^3+1} \cdot x^2 dx = \int_1^2 \frac{1}{u} \cdot \frac{1}{3} du = \frac{1}{3}\int_1^2 \frac{1}{u} du$$

$$= \frac{1}{3} \ln|u| \Big|_1^2 = \frac{1}{3}(\ln 2 - \ln 1) = \frac{1}{3}\ln 2$$

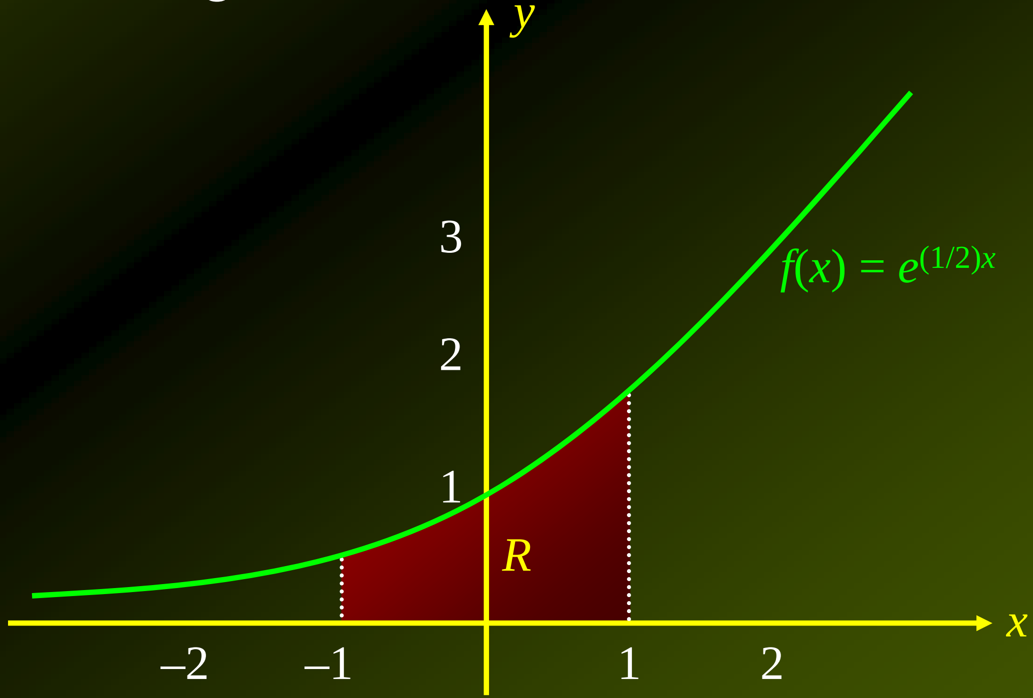
Example 4 – *Using the Method of Substitution*

Find the area of the region R under the graph of

$f(x) = e^{(1/2)x}$ from $x = -1$ to $x = 1$.

Solution:

The graph shows region R:



Example 4 – *Solution*

cont'd

Since $f(x)$ is always greater than zero, the area is given by

$$A = \int_{-1}^{1} e^{(1/2)x}\,dx$$

To evaluate this integral, we substitute

$$u = \frac{1}{2}x$$

so that

$$du = \frac{1}{2}dx$$

$$2du = dx$$

Example 4 – *Solution*

cont'd

When $x = -1$, $u = -\frac{1}{2}$, and when $x = 1$, $u = \frac{1}{2}$.

Substitute $dx = 2du$, along with the limits of integration:

$$A = \int_{-1}^{1} e^{(1/2)x} dx = \int_{-1/2}^{1/2} e^{u} \cdot 2du$$

$$= 2\int_{-1/2}^{1/2} e^{u} du$$

$$= 2e^{u} \Big|_{-1/2}^{1/2}$$

$$= 2(e^{1/2} - e^{-1/2}) \approx 2.08$$

or approximately 2.08 square units.

Average Value of a Function

Suppose *f* is integrable on [*a*, *b*].

Then, the average value of *f* over [*a*, *b*] is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Applied Example 6 – *Automobile Financing*

The interest rates changed by Madison Finance on auto loans for used cars over a certain 6-month period in 2008 are approximated by the function

$$r(t) = -\frac{1}{12}t^3 + \frac{7}{8}t^2 - 3t + 12 \qquad (0 \le t \le 6)$$

where t is measured in months and $r(t)$ is the annual percentage rate.

What is the average rate on auto loans extended by Madison over the 6-month period?

Applied Example 6 – *Solution*

The average rate over the 6-month period is given by

$$\frac{1}{6-0}\int_0^6 \left(-\frac{1}{12}t^3+\frac{7}{8}t^2-3t+12\right)dx$$

$$=\frac{1}{6}\left(-\frac{1}{48}t^4+\frac{7}{24}t^3-\frac{3}{2}t^2+12t\right)\Bigg|_0^6$$

$$=\frac{1}{6}\left(-\frac{1}{48}(6)^4+\frac{7}{24}(6)^3-\frac{3}{2}(6)^2+12(6)\right)$$

$$=9$$

or 9% per year.