6

INTEGRATION

6.6 Area Between Two Curves

The Area Between Two Curves

Let f and g be continuous functions such that $f(x) \geq g(x)$ on the interval $[a, b]$.

Then, the area of the region bounded above by $y = f(x)$ and below by $y = g(x)$ on $[a, b]$ is given by

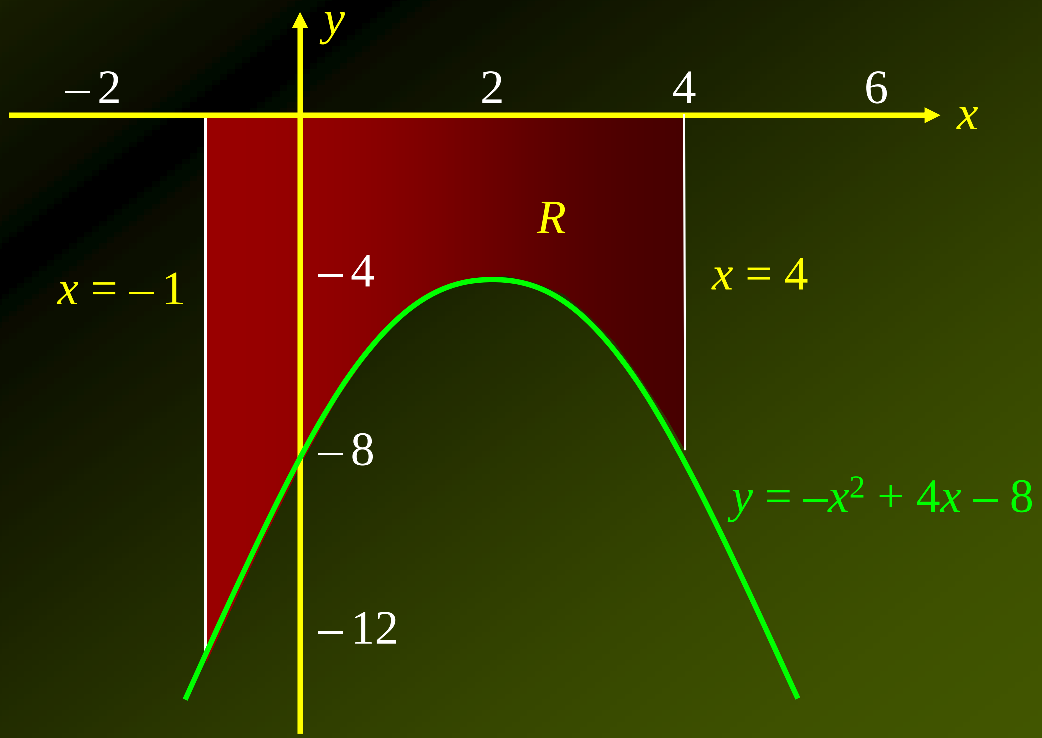
$$\int_a^b [f(x) - g(x)]\,dx$$

Example 1

Find the area of the region bounded by the x-axis, the graph of $y = -x^2 + 4x - 8$, and the lines $x = -1$ and $x = 4$.

Solution:

The region R is being bounded above by the graph $f(x) = 0$ and below by the graph of $g(x) = y = -x^2 + 4x - 8$ on $[-1, 4]$:



Example 1 – *Solution*

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Therefore, the area of *R* is given by

$$\int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{4} \left[0 - (-x^2 + 4x - 8)\right] dx$$

$$= \int_{-1}^{4} (x^2 - 4x + 8)\,dx$$

$$= \frac{1}{3}x^3 - 2x^2 + 8x \Big|_{-1}^{4}$$

$$= \left[\frac{1}{3}(4)^3 - 2(4)^2 + 8(4)\right] - \left[\frac{1}{3}(-1)^3 - 2(-1)^2 + 8(-1)\right]$$

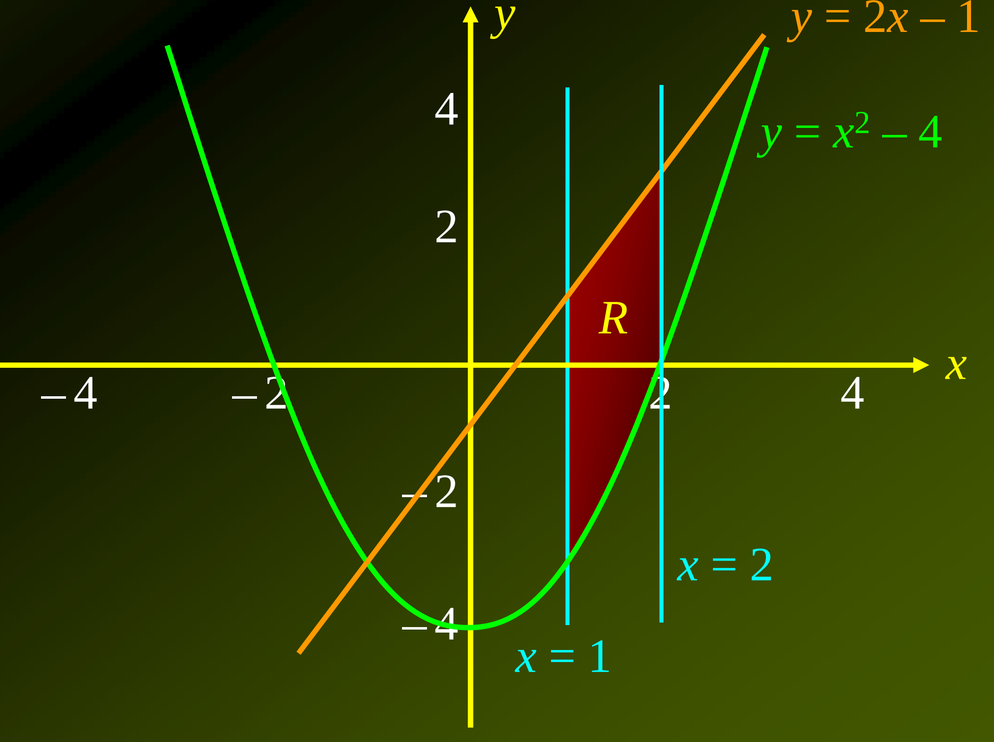
$$= 31\tfrac{2}{3}$$

Example 2

Find the area of the region bounded by $f(x) = 2x - 1$,
$g(x) = x^2 - 4$, $x = 1$, and $x = 2$.

Solution:

Note that the graph of f always lies above that of g for all x in the interval $[1, 2]$:



Example 2 – *Solution*

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Since the graph of *f* always lies above that of *g* for all *x* in the interval [1, 2], the required area is given by

$$\int_a^b \left[f(x) - g(x) \right] dx = \int_1^2 \left[(2x - 1) - (x^2 - 4) \right] dx$$

$$= \int_1^2 (-x^2 + 2x + 3)\,dx = -\frac{1}{3}x^3 + x^2 + 3x \Big|_1^2$$

$$= \left(-\frac{1}{3}(2)^3 + (2)^2 + 3(2) \right) - \left(-\frac{1}{3}(1)^3 + (1)^2 + 3(1) \right)$$

$$= \frac{11}{3}$$

Example 3

Find the area of the region that is completely enclosed by the graphs of $f(x) = 2x – 1$ and $g(x) = x^2 – 4$.

Solution:

First, find the points of intersection of the two curves.
To do this, you can set $g(x) = f(x)$ and solve for x:

$$x^2 - 4 = 2x - 1$$

$$x^2 - 2x - 3 = 0$$

$$(x+1)(x-3) = 0$$

so, the graphs intersect at $x = –1$ and at $x = 3$.

Example 3 – *Solution*

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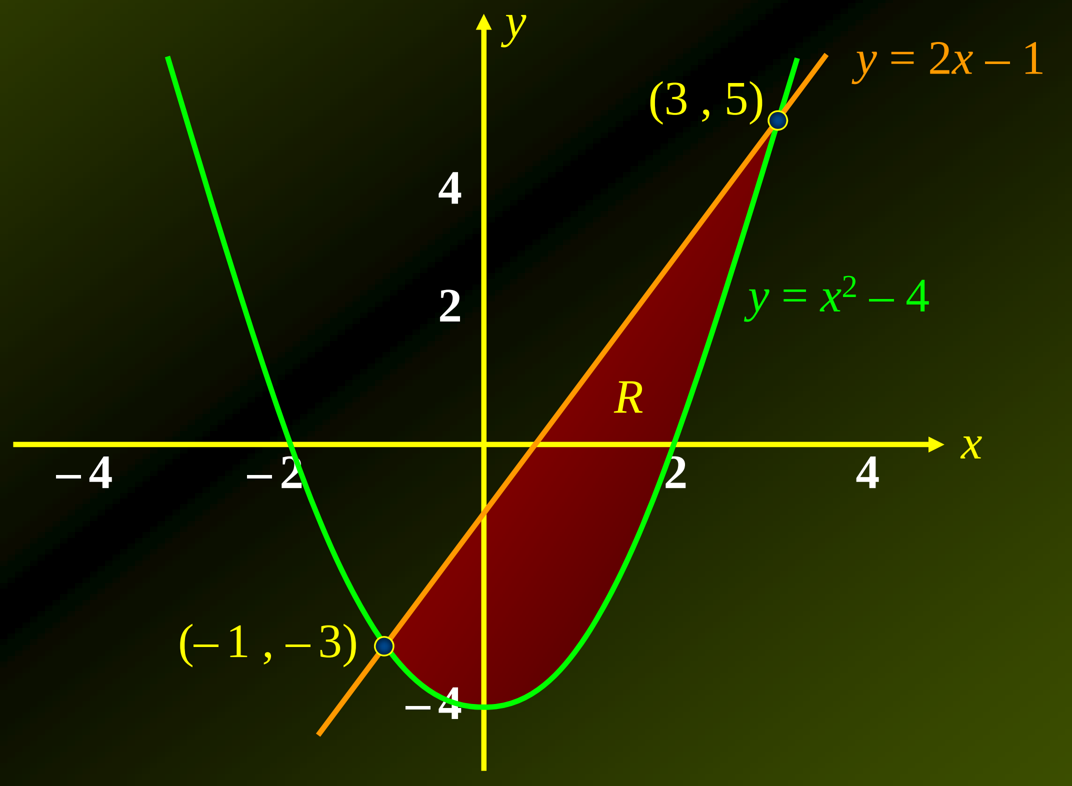
The graph of *f* always lies above that of *g* for all *x* in the interval [–1, 3] between the two intersection points:



Example 3 – *Solution*

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Since the graph of *f* always lies above that of *g* for all *x* in the interval [–1, 3], the required area is given by

$$\int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{3} \left[(2x - 1) - (x^2 - 4)\right] dx$$

$$= \int_{-1}^{3} (-x^2 + 2x + 3)\,dx = -\frac{1}{3}x^3 + x^2 + 3x \Big|_{-1}^{3}$$

$$= \left(-\frac{1}{3}(3)^3 + (3)^2 + 3(3)\right) - \left(-\frac{1}{3}(-1)^3 + (-1)^2 + 3(-1)\right)$$

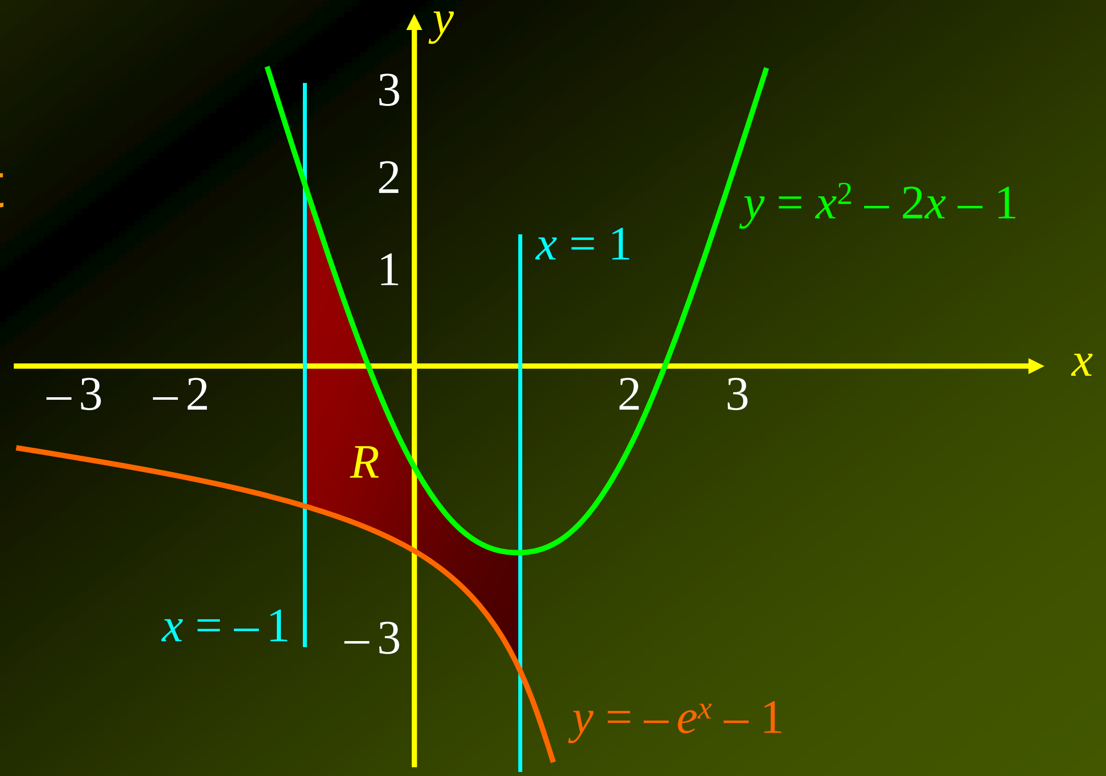
$$= 10\frac{2}{3}$$

Example 4

Find the area of the region bounded by $f(x) = x^2 - 2x - 1$, $g(x) = -e^x - 1$, $x = -1$, and $x = 1$.

Solution:

Note that the graph of f always lies above that of g for all x in the interval [–1, 1]:



Example 4 – *Solution*

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Since the graph of *f* always lies above that of *g* for all *x* in the interval [–1, 1], the required area is given by

$$\int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{1} \left[(x^2 - 2x - 1) - (-e^x - 1)\right] dx$$

$$= \int_{-1}^{1} (x^2 - 2x + e^x)\,dx = \frac{1}{3}x^3 - x^2 + e^x \Big|_{-1}^{1}$$

$$= \left(\frac{1}{3}(1)^3 - (1)^2 + e^{(1)}\right) - \left(\frac{1}{3}(-1)^3 - (-1)^2 + e^{(-1)}\right)$$

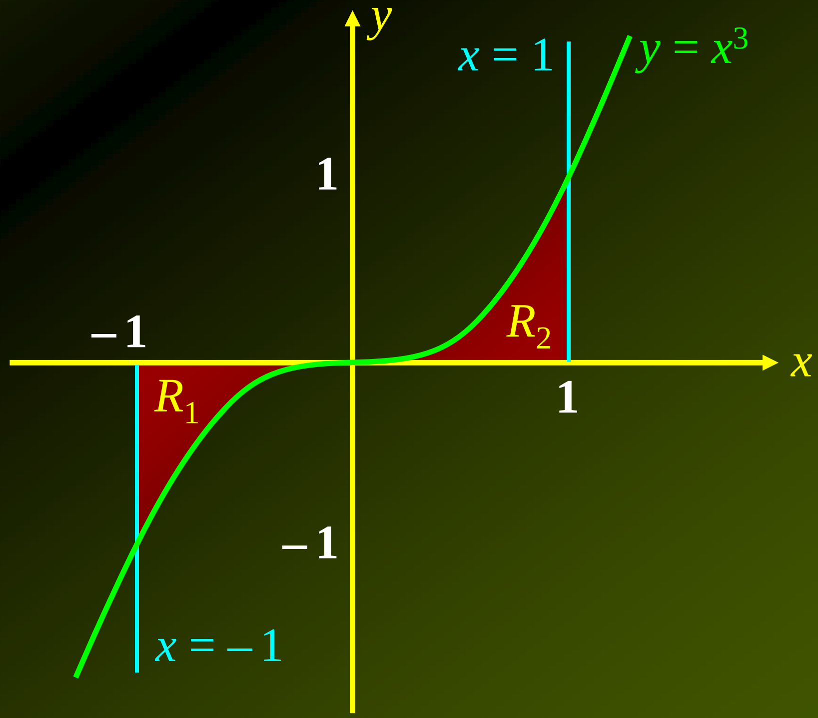
$$= \frac{2}{3} + e - \frac{1}{e} \approx 3.02$$

Example 5

Find the area of the region bounded by $f(x) = x^3$, the x-axis, $x = -1$, and $x = 1$.

Solution:

The region being considered is composed of two subregions R_1 and R_2:



Example 5 – *Solution*

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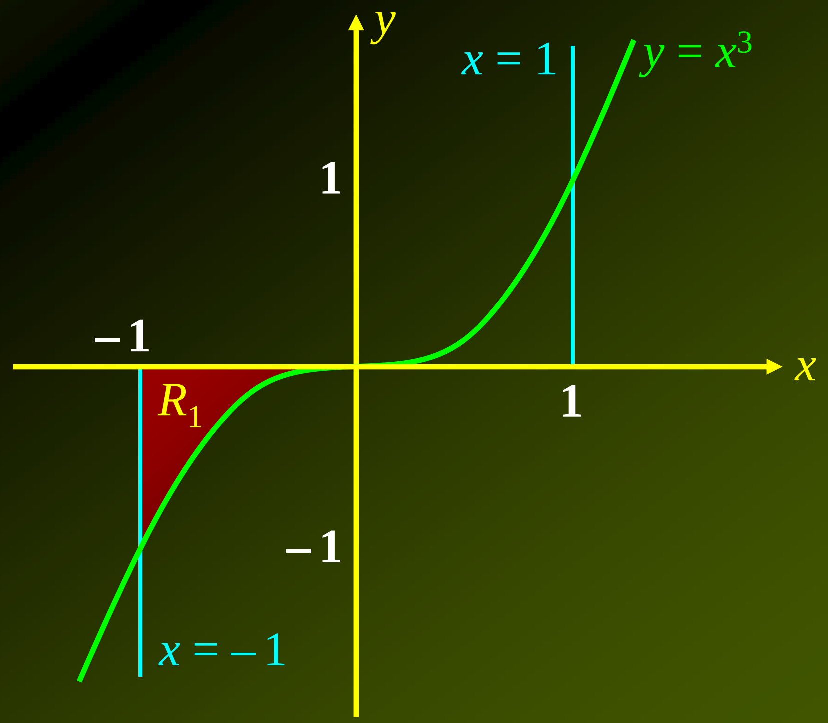
To find R_1 and R_2 consider the x-axis as $g(x) = 0$.

Since $g(x) \geq f(x)$ on $[-1, 0]$, the area of R_1 is given by

$$R_1 = \int_a^b [g(x) - f(x)]\,dx$$

$$= \int_{-1}^{0} (0 - x^3)\,dx = -\int_{-1}^{0} x^3\,dx$$

$$= -\frac{1}{4}x^4\Big|_{-1}^{0} = 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$$



Example 5 – *Solution*

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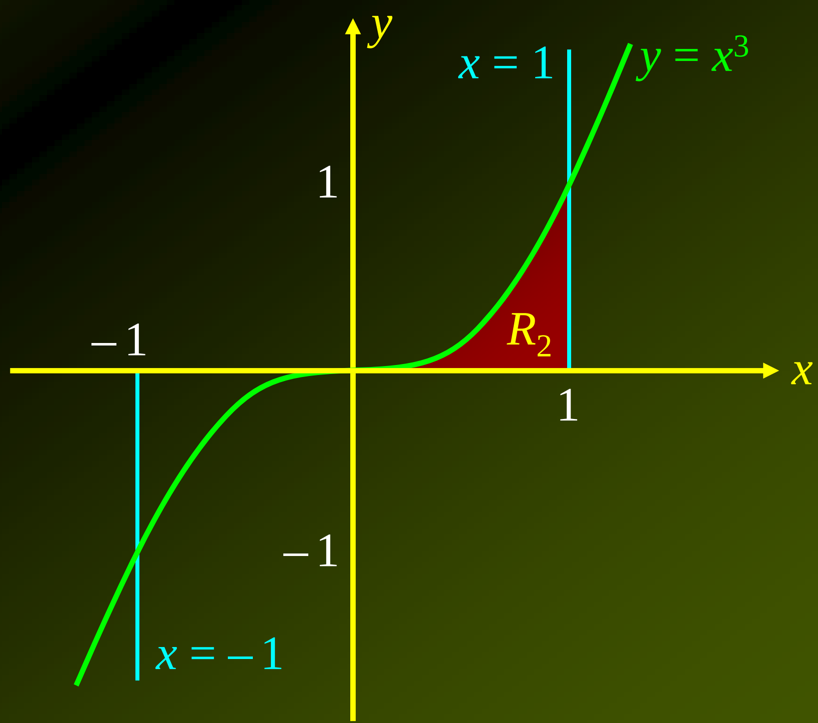
To find R_1 and R_2 consider the x-axis as $g(x) = 0$.

Since $g(x) \leq f(x)$ on [0, 1], the area of R_2 is given by

$$R_2 = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_0^1 (x^3 - 0)\,dx = \int_0^1 x^3\,dx$$

$$= \frac{1}{4}x^4 \Big|_0^1 = \left(\frac{1}{4}\right) - 0 = \frac{1}{4}$$



Example 5 – *Solution*

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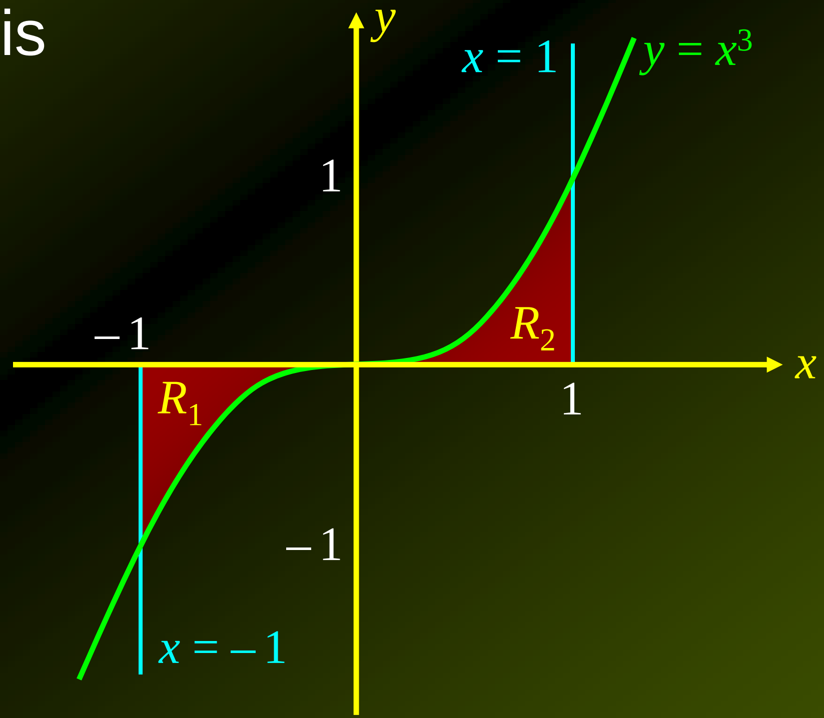
Therefore, the required area *R* is

$$R = R_1 + R_2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

square units.

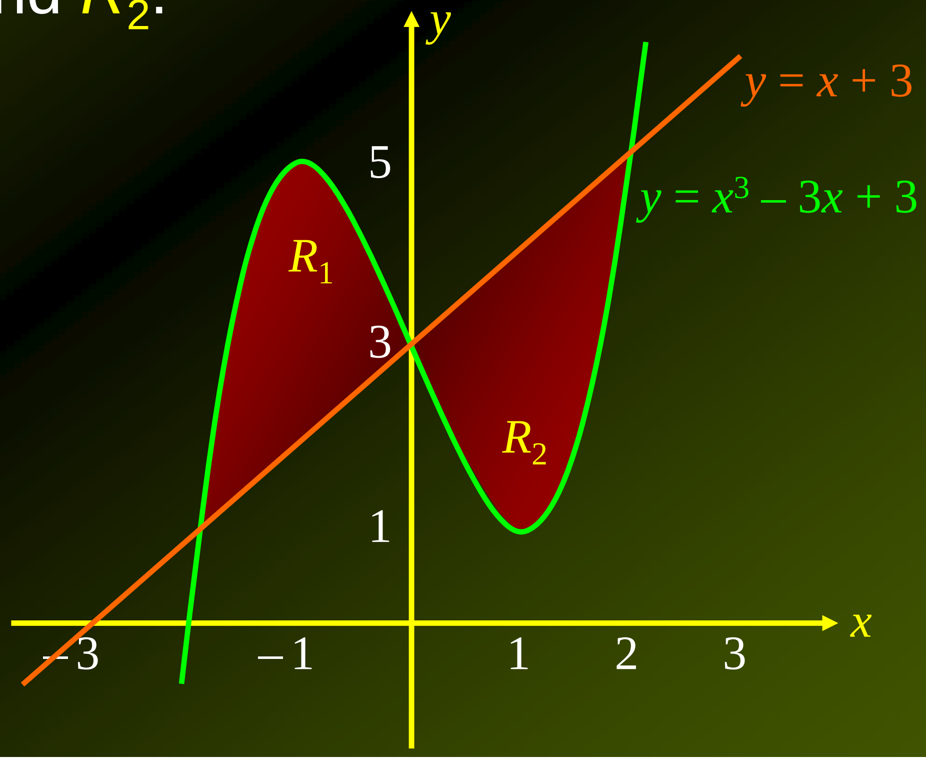


Example 6

Find the area of the region bounded by $f(x) = x^3 - 3x + 3$ and $g(x) = x + 3$.

Solution:

The region R being considered is composed of two subregions R_1 and R_2:



Example 6 – *Solution*

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To find the points of intersection, we solve simultaneously the equations $y = x^3 - 3x + 3$ and $y = x + 3$.

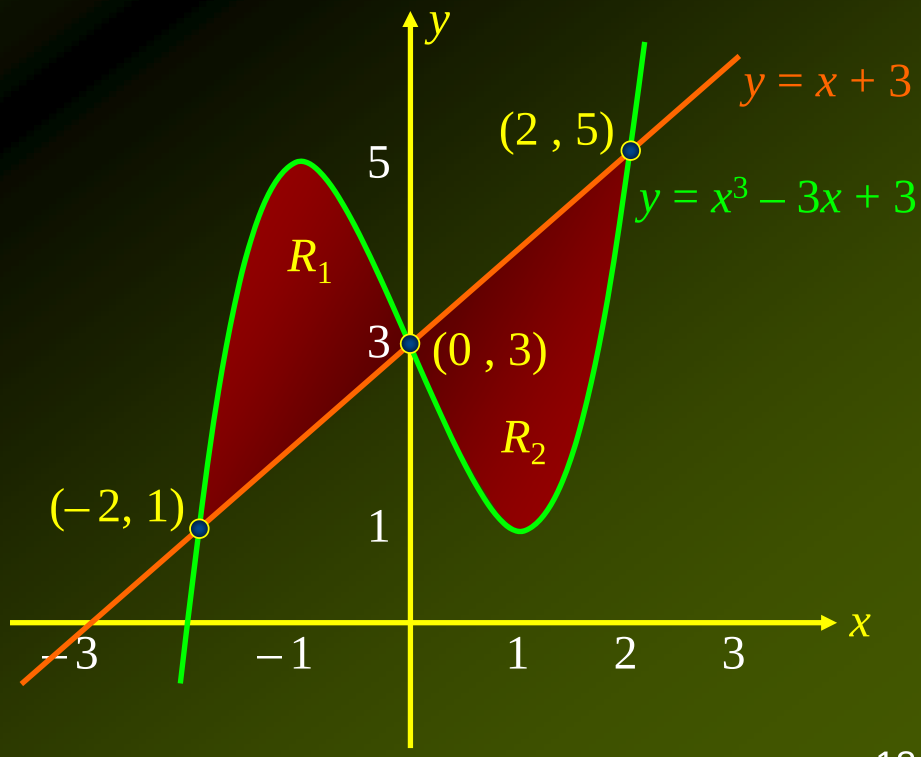
$$x^3 - 3x + 3 = x + 3$$

$$x^3 - 4x = 0$$

$$x(x + 2)(x - 2) = 0$$

So, $x = 0$, $x = -2$, and $x = 2$.

The points of intersection of the two curves are $(-2, 1)$, $(0, 3)$, and $(2, 5)$.



Example 6 – *Solution*

cont'd

Note that $f(x) \geq g(x)$ for $[-2, 0]$, so the area of region R_1 is given by

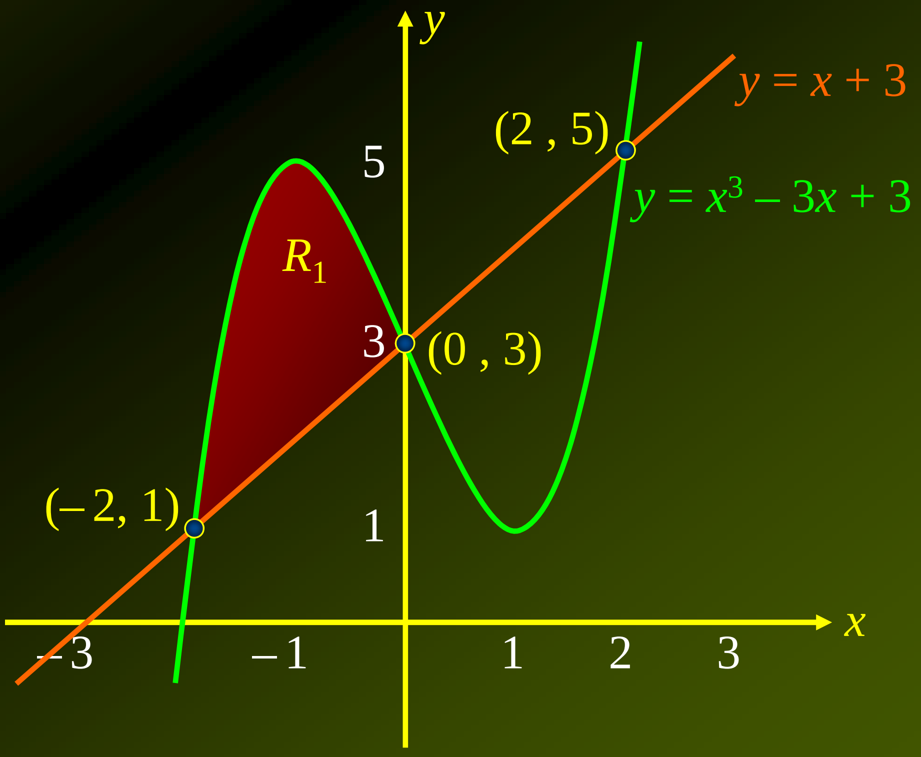
$$R_1 = \int_a^b [f(x) - g(x)]\,dx$$

$$= \int_{-2}^{0} \left[(x^3 - 3x + 3) - (x + 3)\right] dx$$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx$$

$$= \frac{1}{4}x^4 - 2x^2 \Big|_{-2}^{0}$$

$$= -(4 - 8) = 4$$



Example 6 – *Solution*

cont'd

Note that $g(x) \geq f(x)$ for [0, 2], so the area of region R_2 is given by

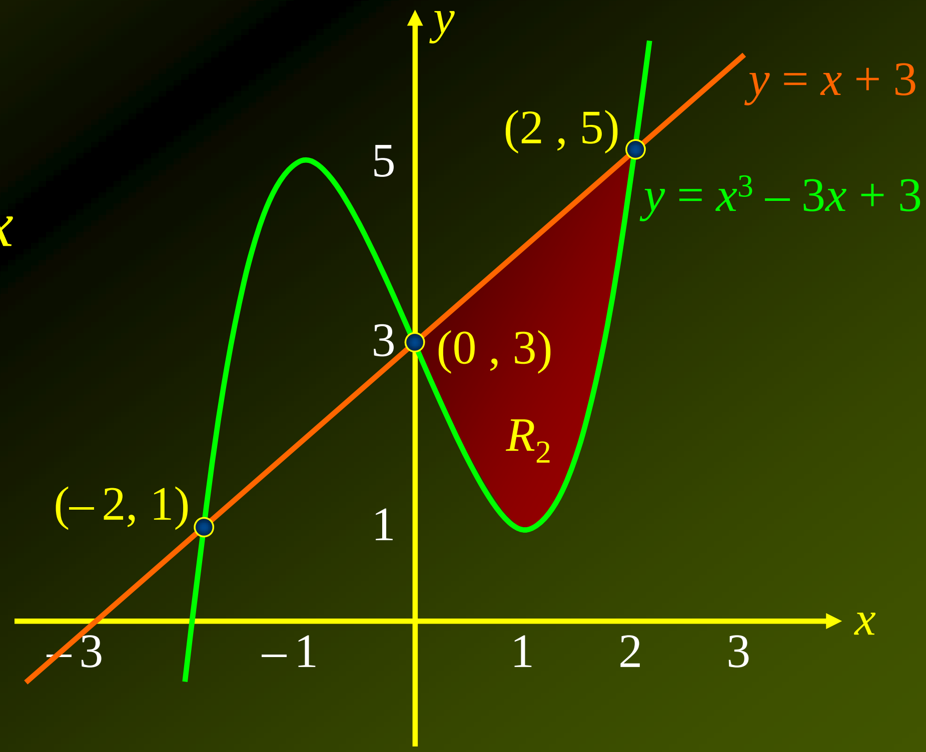
$$R_2 = \int_a^b [g(x) - f(x)]\,dx$$

$$= \int_0^2 \left[(x+3) - (x^3 - 3x + 3)\right] dx$$

$$= \int_0^2 (-x^3 + 4x)\,dx$$

$$= -\frac{1}{4}x^4 + 2x^2 \Big|_0^2$$

$$= -4 + 8 = 4$$



Example 6 – *Solution*

cont'd

Therefore, the required area *R* is

$$R = R_1 + R_2 = 4 + 4 = 8$$

square units.