6

INTEGRATION

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6.7 Applications of the Definite Integral to Business and Economics

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Consumers' and Producers' Surplus

Suppose $p = D(x)$ is the demand function that relates the price p of a commodity to the quantity x demanded of it.

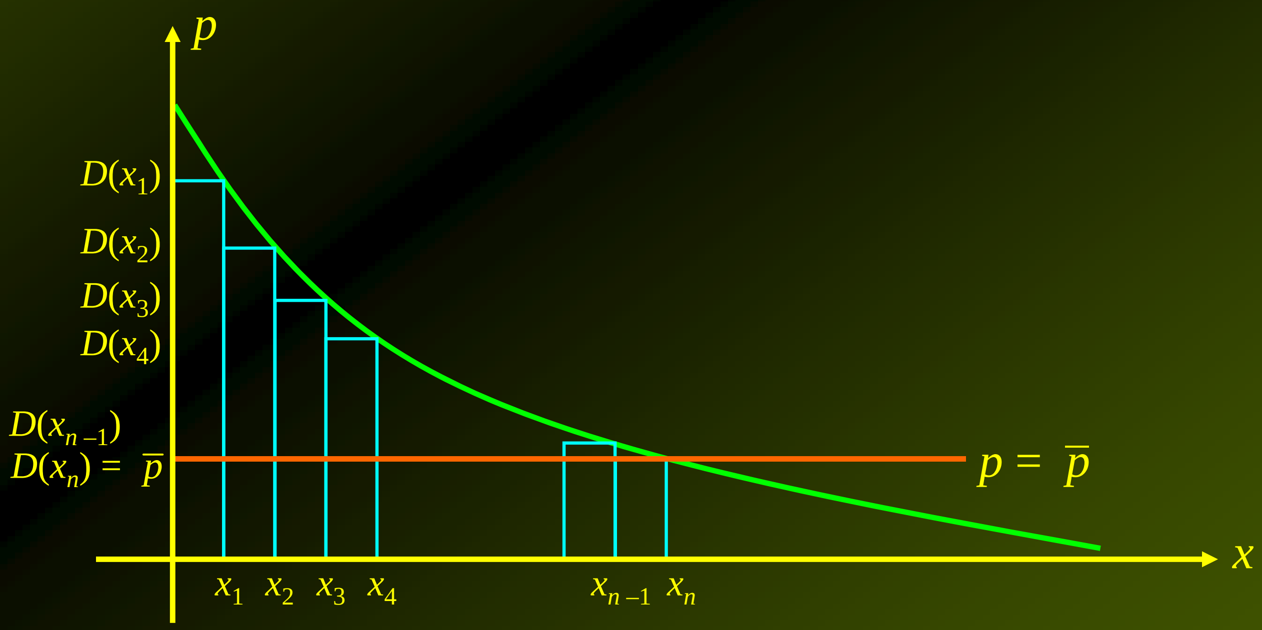
Now suppose a unit market price $\bar{p}$ has been established, along with a corresponding quantity demanded $\bar{x}$.

Those consumers who would be willing to pay a unit price higher than $\bar{p}$ for the commodity would in effect experience a savings.

This difference between what the consumer would be *willing* to pay and what they actually *have* to pay is called the consumers' surplus.

Consumers' and Producers' Surplus

To derive a formula for computing the consumers' surplus, divide the interval $[0, \bar{x}]$ into n subintervals, each of length $\Delta x = \bar{x}/n$, and denote the right endpoints of these intervals by $x_1, x_2, \ldots, x_n = \bar{x}$:



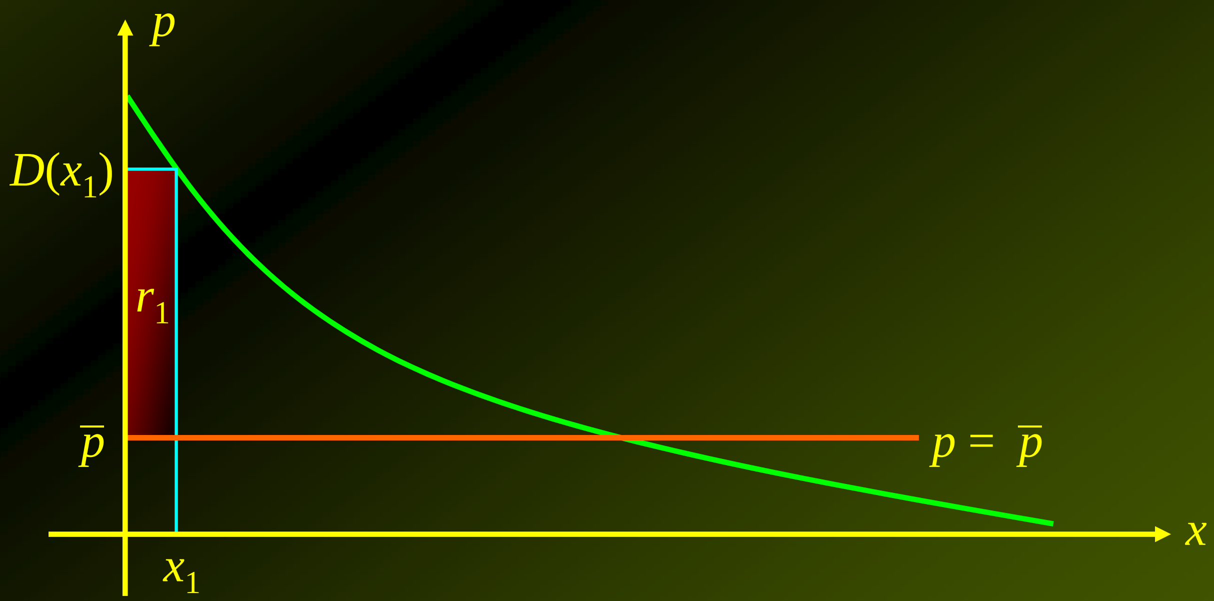
Consumers' and Producers' Surplus

There are consumers who would pay a price of at least $D(x_1)$ for the first Δx units instead of the market price of $\bar{p}$.

The savings to these consumers is approximated by

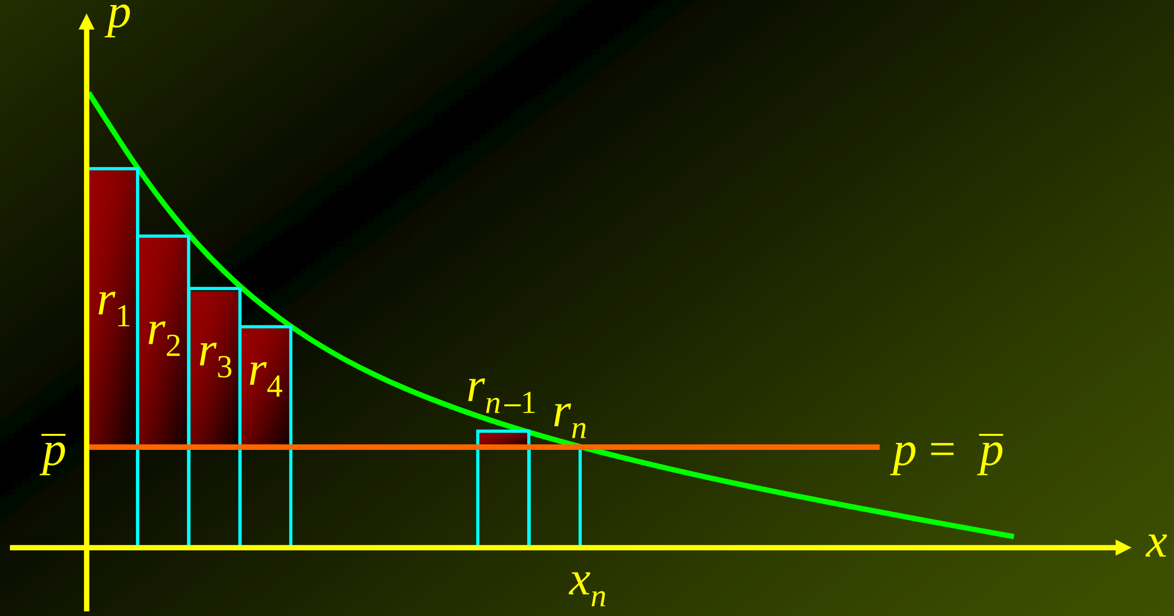
$$D(x_1)\Delta x - \bar{p}\Delta x = [D(x_1) - \bar{p}]\Delta x$$

which is the area of the rectangle r_1:



Consumers' and Producers' Surplus

Similarly, the savings the consumer experiences for the consecutive increments of Δx are depicted by the areas of rectangles r_2, r_3, r_4, … , r_n:

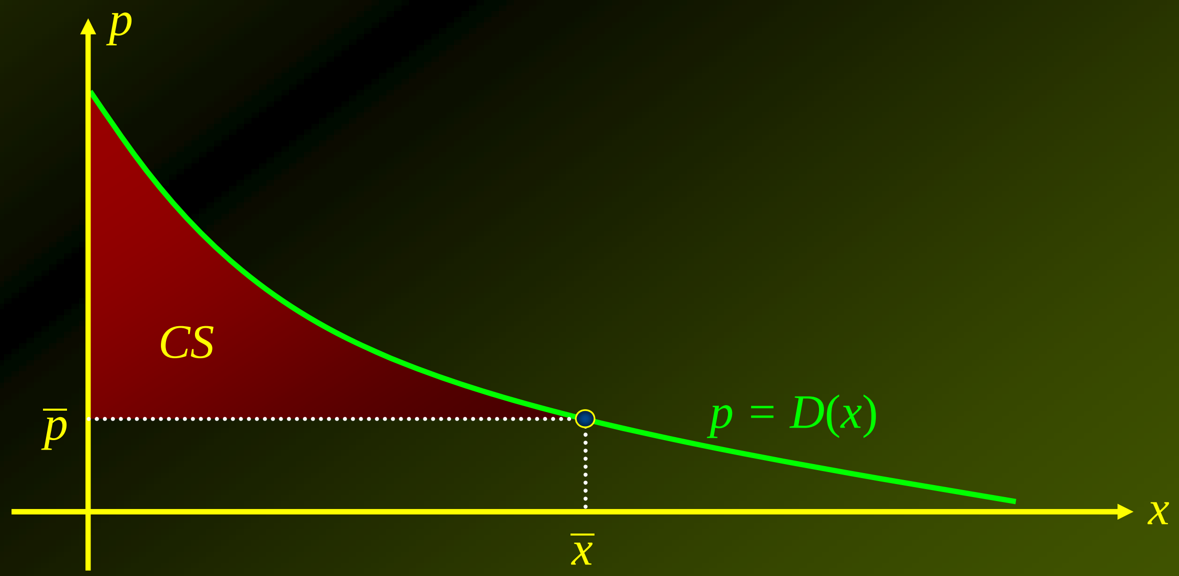


Consumers' and Producers' Surplus

Adding $r_1 + r_2 + r_3 + \ldots + r_n$, and letting n approach infinity, we obtain the consumers' surplus CS formula:

$$CS = \int_0^{\bar{x}} D(x)dx - \bar{p}\,\bar{x}$$

where $D(x)$ is the demand function, $\bar{p}$ is the unit market price, and $\bar{x}$ is the quantity demanded.



Consumers' and Producers' Surplus

Similarly, we can derive a formula for the producers' surplus. Suppose $p = S(x)$ is the supply function that relates the price p of a commodity to the quantity x supplied of it.

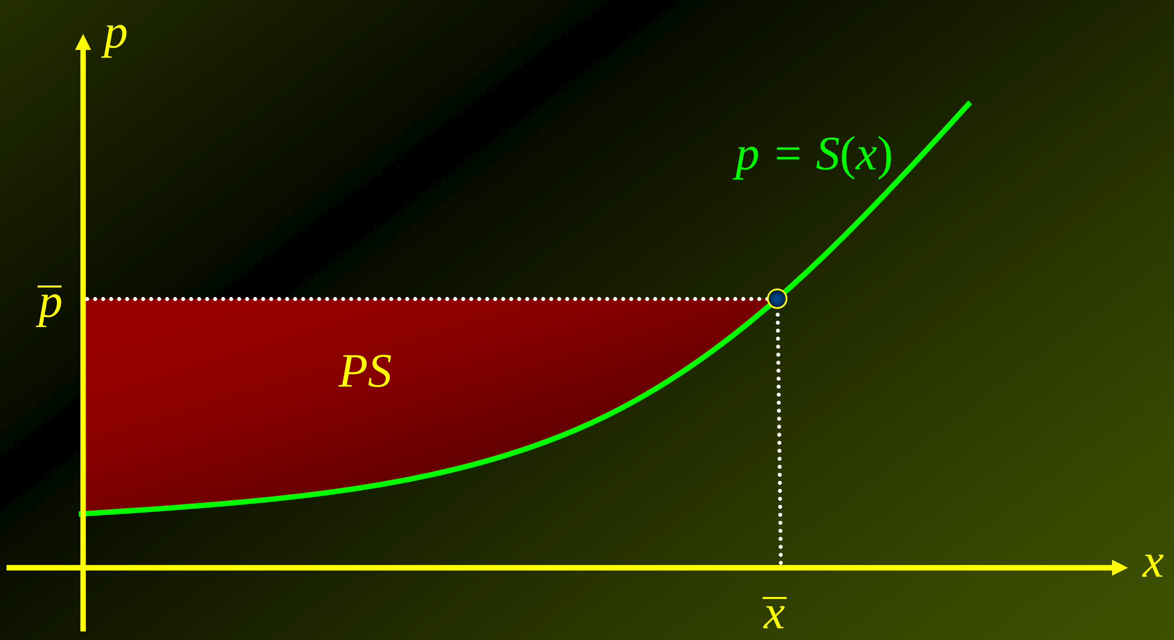
Again, suppose a unit market price $\bar{p}$ has been established, along with a corresponding quantity supplied $\bar{x}$.

Those sellers who would be willing to sell at unit price lower than $\bar{p}$ for the commodity would in effect experience a gain or profit.

This difference between what the seller would be *willing* to sell for and what they actually *can* sell for is called the producers' surplus.

Consumers' and Producers' Surplus

Geometrically, the producers' surplus is given by the area of the region bounded above the straight line $p = \bar{p}$ and below the supply curve $p = S(x)$ from $x = 0$ to $x = \bar{x}$:

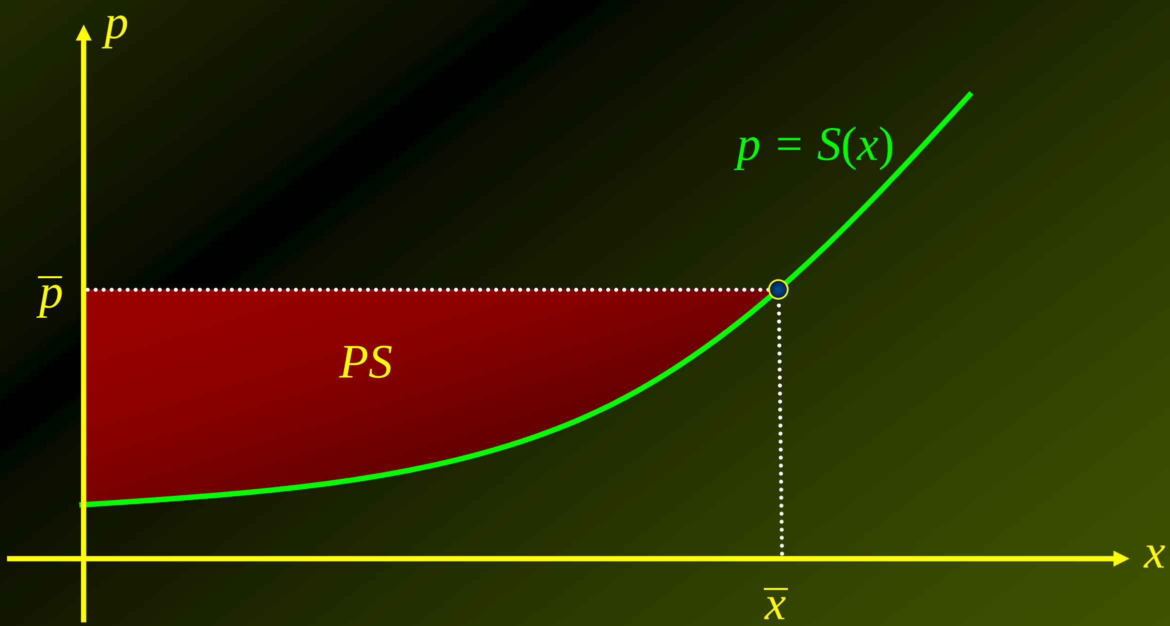


Consumers' and Producers' Surplus

The producers' surplus *PS* is given by

$$PS = \bar{p}\,\bar{x} - \int_0^{\bar{x}} S(x)\,dx$$

where $S(x)$ is the supply function, $\bar{p}$ is the unit market price, and $\bar{x}$ is the quantity supplied.



Example 1

The demand function for a certain make of 10-speed bicycle is given by

$$p = D(x) = -0.001x^2 + 250$$

where p is the unit price in dollars and x is the quantity demanded in units of a thousand.

The supply function for these bicycles is given by

$$p = S(x) = 0.0006x^2 + .02x + 100$$

where p stands for the price in dollars and x stands for the number of bicycles that the supplier will want to sell.

Determine the consumers' surplus and the producers' surplus if the market price of a bicycle is set at the equilibrium price.

Example 1 – *Solution*

To find the equilibrium point, equate $S(x)$ and $D(x)$ to solve the system of equations and find the point of intersection of the demand and supply curves:

$$0.0006x^2 + .02x + 100 = -0.001x^2 + 250$$

$$0.0016x^2 + .02x - 150 = 0$$

$$16x^2 + 200x - 1{,}500{,}000 = 0$$

$$2x^2 + 25x - 187{,}500 = 0$$

$$(2x + 625)(x - 300) = 0$$

Thus, $x = -625/2$ or $x = 300$.

Example 1 – *Solution*

cont'd

The first number is discarded for being negative, so the solution is $x = 300$.

Substitute $x = 300$ to find the equilibrium value of p:

$$p = -0.001(300)^2 + 250 = 160$$

Thus, the equilibrium point is (300, 160).

That is, the equilibrium quantity is 300,000 bicycles, and the equilibrium price is $160 per bicycle.

Example 1 – *Solution*

cont'd

To find the consumers' surplus, we set $\bar{x} = 300$ and $\bar{p} = 160$ in the consumers' surplus formula:

$$CS = \int_0^{\bar{x}} D(x)dx - \bar{p}\,\bar{x}$$

$$= \int_0^{300} (-0.001x^2 + 250)dx - (160)(300)$$

$$= \left(-\frac{1}{3000}x^3 + 250x \right)\Bigg|_0^{300} - 48,000$$

$$= -\frac{300^3}{3000} + 250(300) - 48,000 = 18,000$$

or $18,000,000.

Example 1 – *Solution*

cont'd

To find the producers' surplus, we set $\bar{x}$ = 300 and $\bar{p}$ = 160 in the producers' surplus formula:

$$PS = \bar{p}\,\bar{x} - \int_0^{\bar{x}} S(x)\,dx$$

$$= (160)(300) - \int_0^{300} (0.0006x^2 + 0.02x + 100)\,dx$$

$$= 48,000 - (0.0002x^3 + 0.01x^2 + 100x)\Big|_0^{300}$$

$$= 48,000 - [0.0002(300)^3 + 0.01(300)^2 + 100(300)]$$

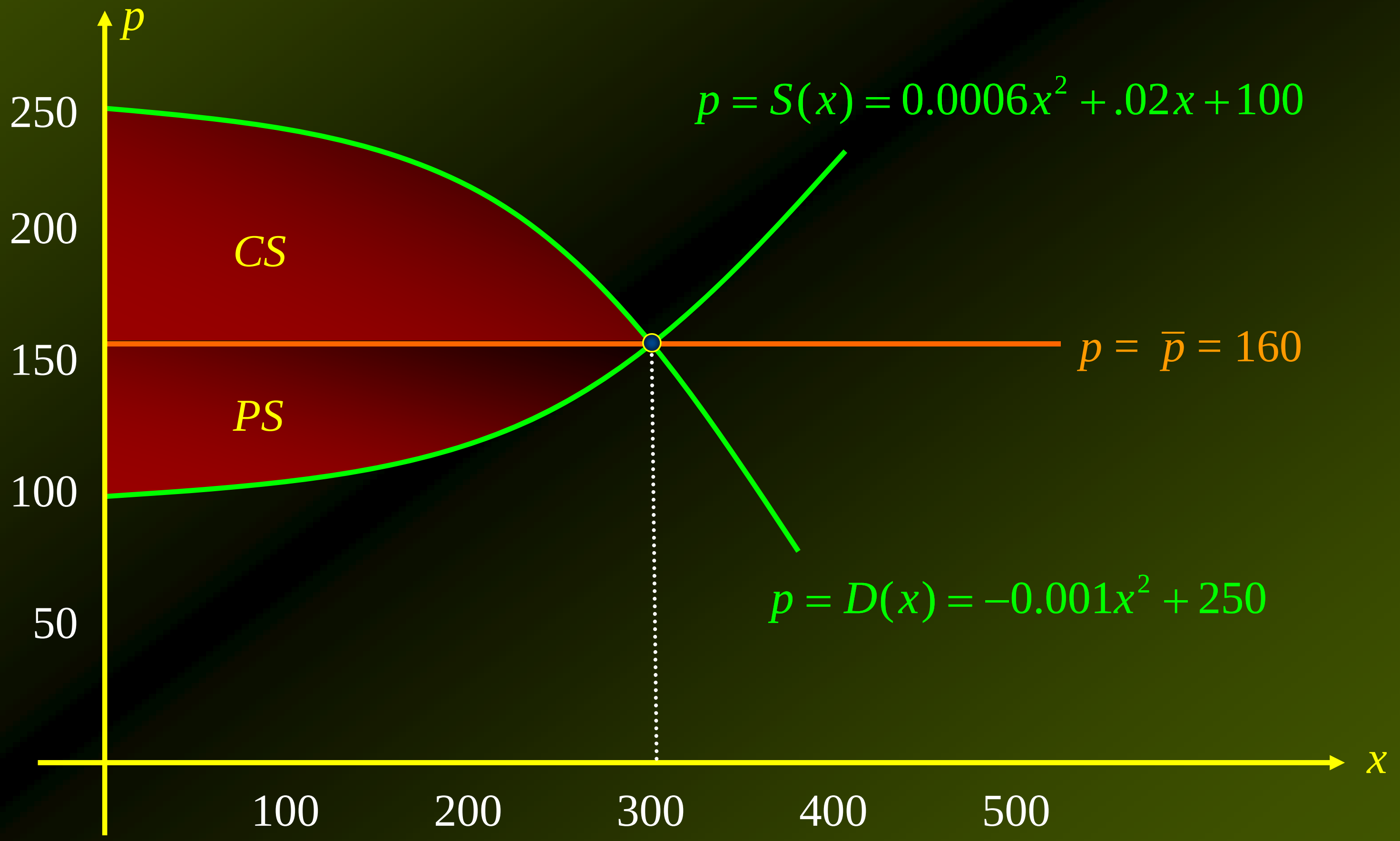
$$= 11,700$$

or $11,700,000.

Example 1 – *Solution*

cont'd

Consumers' surplus and producers' surplus when the market is in equilibrium:



Accumulated or Total Future Value of an Income Stream

The accumulated, or total, future value after T years of an income stream of $R(t)$ dollars per year, earning interest rate of r per year compounded continuously, is given by

$$A = e^{rT}\int_0^T R(t)e^{-rt}\,dt$$

Applied Example 2 – *Income Stream*

Crystal Car Wash recently bought an automatic car-washing machine that is expected to generate $40,000 in revenue per year, t years from now, for the next 5 years. If the income is reinvested in a business earning interest at the rate of 12% per year compounded continuously, find the total accumulated value of this income stream at the end of 5 years.

Applied Example 2 – *Solution*

We are required to find the total future value of the given income stream after 5 years.

Setting $R(t) = 40{,}000$, $r = 0.12$, and $T = 5$ in the accumulated income stream formula we get

$$A = e^{rT}\int_0^T R(t)e^{-rt}\,dt = e^{0.12(5)}\int_0^5 40{,}000e^{-0.12t}\,dt$$

$$= e^{0.6}\left[-\frac{40{,}000}{0.12}e^{-0.12t}\right]\Bigg|_0^5$$

$$= -\frac{40{,}000e^{0.6}}{0.12}(e^{-0.6}-1) \approx 274{,}039.60$$

or approximately $274,040.

Present Value of an Income Stream

The present value of an income stream of $R(t)$ dollars in a year, earning interest at the rate of r per year compounded continuously, is given by

$$PV = \int_0^T R(t)e^{-rt}dt$$

Applied Example 2 – *Investment Analysis*

The owner of a local cinema is considering two alternative plans for renovating and improving the theater.

Plan A calls for an immediate cash outlay of $250,000, whereas plan B requires an immediate cash outlay of $180,000.

It has been estimated that adopting plan A would result in a net income stream generated at the rate of

$$f(t) = 630{,}000$$

dollars per year, whereas adopting plan B would result in a net income stream generated at the rate of

$$g(t) = 580{,}000$$

for the next three years.

Applied Example 2 – *Investment Analysis* cont'd

If the prevailing interest rate for the next five years is 10% per year, which plan will generate a higher net income by the end of year 3?

Solution:

We can find the present value of the net income *NI* for plan A setting $R(t) = 630{,}000$, $r = 0.1$, and $T = 3$, using the present value formula:

$$NI = \int_0^T R(t)e^{-rt}\,dt - 250{,}000$$

$$= \int_0^3 630{,}000e^{-0.1t}\,dt - 250{,}000$$

Applied Example 2 – *Solution*

cont'd

$$= \frac{630,000}{-0.1} e^{-0.1t} \Big|_0^3 - 250,000$$

$$= -6,300,000e^{-0.3} + 6,300,000 - 250,000$$

$$\approx 1,382,845$$

or approximately $1,382,845.

To find the present value of the net income *NI* for plan B setting $R(t) = 580,000$, $r = 0.1$, and $T = 3$, using the present value formula:

$$NI = \int_0^T R(t)e^{-rt}dt - 180,000$$

Applied Example 2 – *Solution*

cont'd

$$= \int_0^3 580{,}000e^{-0.1t}\,dt - 180{,}000$$

$$= \frac{580{,}000}{-0.1}e^{-0.1t}\bigg|_0^3 - 180{,}000$$

$$= -5{,}800{,}000e^{-0.3} + 5{,}800{,}000 - 180{,}000$$

$$\approx 1{,}323{,}254$$

or approximately $1,323,254.

Thus, we conclude that plan A will generate a higher present value of net income by the end of the third year ($1,382,845), than plan B ($1,323,254).

Amount of an Annuity

The amount of an annuity is

$$A = \frac{mP}{r}(e^{rT} - 1)$$

where P, r, T, and m are as defined earlier.

Applied Example 4 – *IRAs*

On January 1, 1990, Marcus Chapman deposited $2000 into an Individual Retirement Account (IRA) paying interest at the rate of 10% per year compounded continuously. Assuming that he deposited $2000 annually into the account, how much did he have in his IRA at the beginning of 2006?

Applied Example 4 – *Solution*

We set $P = 2000$, $r = 0.1$, $T = 16$, and $m = 1$ in the amount of annuity formula, obtaining

$$A = \frac{mP}{r}(e^{rT} - 1) = \frac{2000}{0.1}(e^{1.6} - 1)$$

$$\approx 79{,}060.65$$

Thus, Marcus had approximately $79,061 in his account at the beginning of 2006.

Present Value of an Annuity

The present value of an annuity is given by

$$PV = \frac{mP}{r}(1 - e^{-rT})$$

where P, r, T, and m are as defined earlier.

Applied Example 5 – *Sinking Funds*

Tomas Perez, the proprietor of a hardware store, wants to establish a fund from which he will withdraw $1000 per month for the next ten years. If the fund earns interest at a rate of 6% per year compounded continuously, how much money does he need to establish the fund?

Applied Example 5 – *Solution*

We want to find the present value of an annuity with $P = 1000$, $r = 0.06$, $T = 10$, and $m = 12$.

Using the present value of an annuity formula, we find

$$PV = \frac{mP}{r}(1 - e^{-rT}) = \frac{12,000}{0.06}(1 - e^{-(0.06)(10)})$$

$$\approx 90,237.70$$

Thus, Tomas needs approximately $90,238 to establish the fund.