7 ADDITIONAL TOPICS IN INTEGRATION

7.1 Integration by Parts

The Method of Integration by Parts

Integration by parts formula

$$\int u \, dv = uv - \int v \, du$$

Example 1

Evaluate $\int xe^x dx$

Solution:

Let $u = x$ and $dv = e^x dx$

So that $du = dx$ and $v = e^x$

Therefore,

$$\int xe^x dx = \int u dv$$
$$= uv - \int v du$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$
$$= (x-1)e^x + C$$

Guidelines for Integration by Parts

Choose u an dv so that

1. du is simpler than u.
2. dv is easy to integrate.

Example 2

Evaluate $\int x \ln x \, dx$

Solution:

Let $u = \ln x$ and $dv = x \, dx$

So that $du = \frac{1}{x} dx$ and $v = \frac{1}{2} x^2$

Therefore,

$$\int x \ln x \, dx = \int u dv$$

$$= uv - \int v du$$

Example 2 – *Solution*

cont'd

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x^2 \cdot \left(\frac{1}{x}\right) dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

$$= \frac{1}{4}x^2 (2 \ln x - 1) + C$$

Example 3

Evaluate $\int \frac{xe^x}{(x+1)^2}\,dx$

Solution:

Let $u = xe^x$ and $dv = \frac{1}{(x+1)^2}dx$

So that $du = (xe^x + e^x)dx$ and $v = -\frac{1}{x+1}$

$$= e^x(x+1)dx$$

Therefore,

$$\int \frac{xe^x}{(x+1)^2}\,dx = \int u\,dv$$

$$= uv - \int v\,du$$

Example 3 – *Solution*

cont'd

$$= xe^x\left(\frac{-1}{x+1}\right) - \int\left(\frac{-1}{x+1}\right)e^x(x+1)dx$$

$$= -\frac{xe^x}{x+1} + \int e^x dx$$

$$= -\frac{xe^x}{x+1} + e^x + C$$

Example 4

Evaluate $\int x^2 e^x \, dx$

Solution:

Let $u = x^2$ and $dv = e^x \, dx$

So that $du = 2x dx$ and $v = e^x$

Therefore,

$$\int x^2 e^x \, dx = \int u dv$$

$$= uv - \int v du$$

Example 4 – *Solution*

cont'd

$$= x^2e^x - \int e^x(2x)dx$$

$$= x^2e^x - 2\int xe^x dx$$

$$= xe^x - 2\left[(x-1)e^x\right] + C$$

(From first example)

$$= e^x(x^2 - 2x + 2) + C$$

Applied Example 5 – *Oil Production*

The estimated rate at which oil will be produced from an oil well *t* years after production has begun is given by

$$R(t) = 100te^{-0.1t}$$

thousand barrels per year.

Find an expression that describes the total production of oil at the end of year *t*.

Applied Example 5 – *Solution*

Let $T(t)$ denote the total production of oil from the well at the end of year t ($t \geq 0$).

Then, the rate of oil production will be given by $T'(t)$ thousand barrels per year.

Thus,

$$T'(t) = R(t) = 100te^{-0.1t}$$

So,

$$T(t) = \int 100te^{-0.1t}\,dt$$

$$= 100\int te^{-0.1t}\,dt$$

Applied Example 5 – *Solution*

cont'd

Use integration by parts to evaluate the integral.

Let $u = t$ and $dv = e^{-0.1t}dt$

So that $du = dt$ and $v = -\frac{1}{0.1}e^{-0.1t}$

$$= -10e^{-0.1t}$$

Therefore,

$$T(t) = 100\int te^{-0.1t}dt = 100\left[-10te^{-0.1t} + 10\int e^{-0.1t}dt\right]$$

$$= 100\left[-10te^{-0.1t} - 100e^{-0.1t}\right] + C$$

$$= -1000e^{-0.1t}(t+10) + C$$

Applied Example 5 – *Solution*

cont'd

To determine the value of C, note that the total quantity of oil produced at the end of year 0 is nil, so $T(0) = 0$.

This gives,

$$T(t) = -1000e^{-0.1t}(t+10) + C$$

$$T(0) = -1000e^{-0.1(0)}(0+10) + C = 0$$

$$= -1000(10) + C = 0$$

$$C = 10,000$$

Thus, the required production function is given by

$$T(t) = -1000e^{-0.1t}(t+10) + 10,000$$