7 ADDITIONAL TOPICS IN INTEGRATION

7.3 Numerical Integration

Approximating Definite Integrals

Sometimes, it is necessary to evaluate definite integrals based on empirical data where there is no algebraic rule defining the integrand.

Other situations also arise in which an integrable function has an antiderivative that cannot be found in terms of elementary functions.

Approximating Definite Integrals

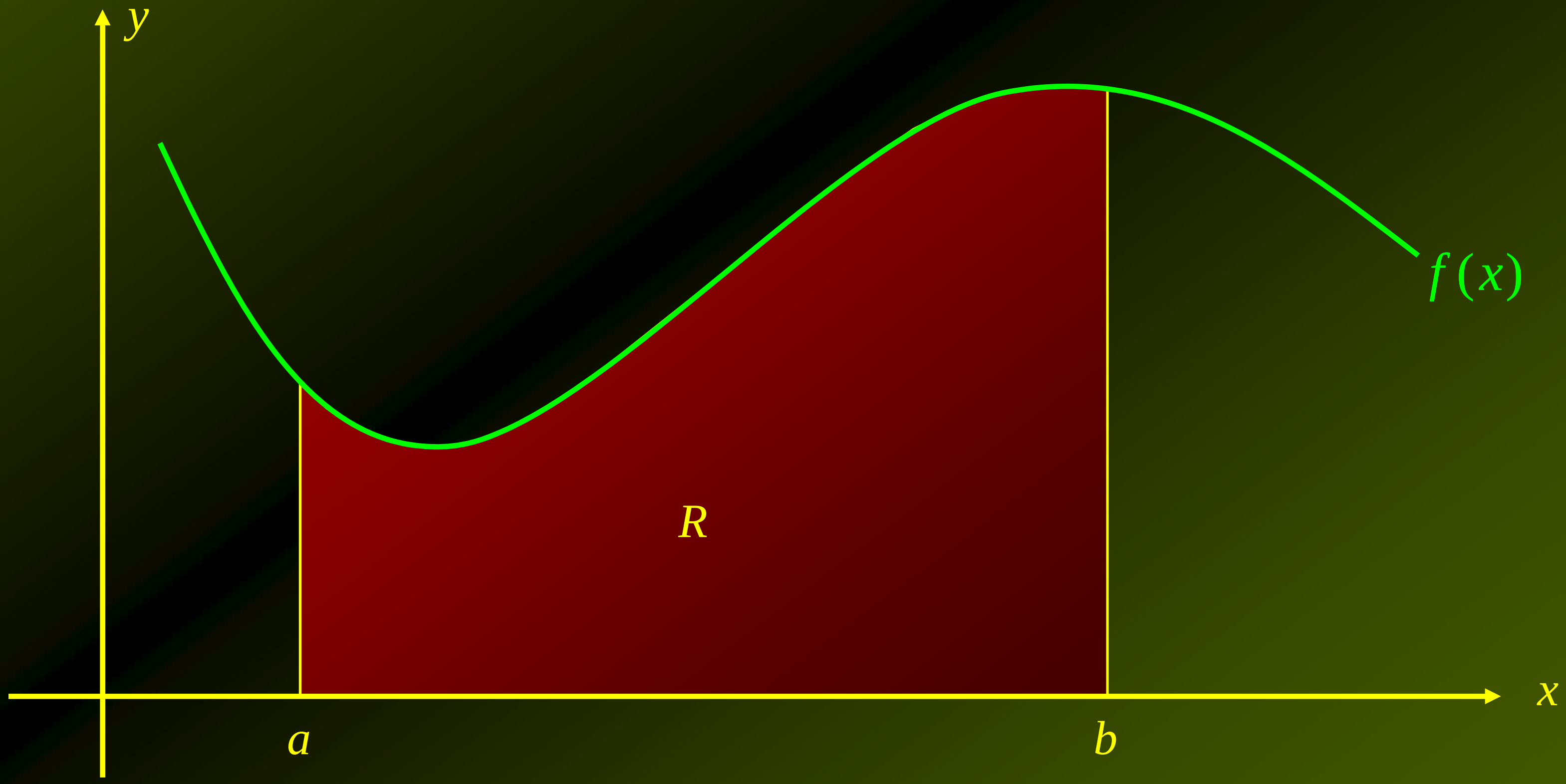
Examples of these are

$$f(x) = e^{x^2} \qquad g(x) = x^{-1/2} e^x \qquad h(x) = \frac{1}{\ln x}$$

Riemann sums provide us with a good approximation of a definite integral, but there are better techniques and formulas, called *quadrature formulas*, that allow a more efficient way of computing approximate values of definite integrals.

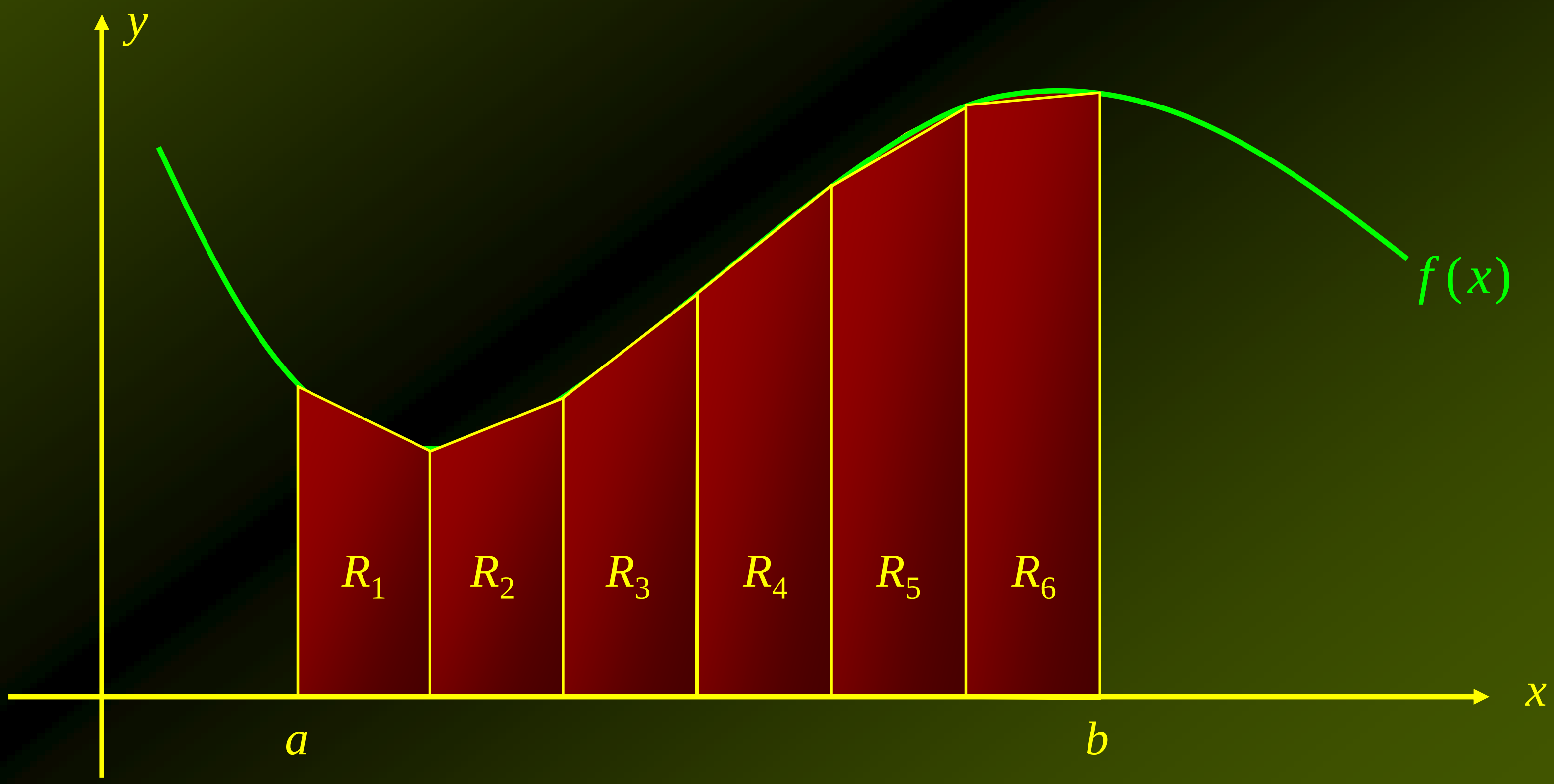
The Trapezoidal Rule

Consider the problem of finding the area under the curve of *f*(*x*) for the interval [*a*, *b*]:



The Trapezoidal Rule

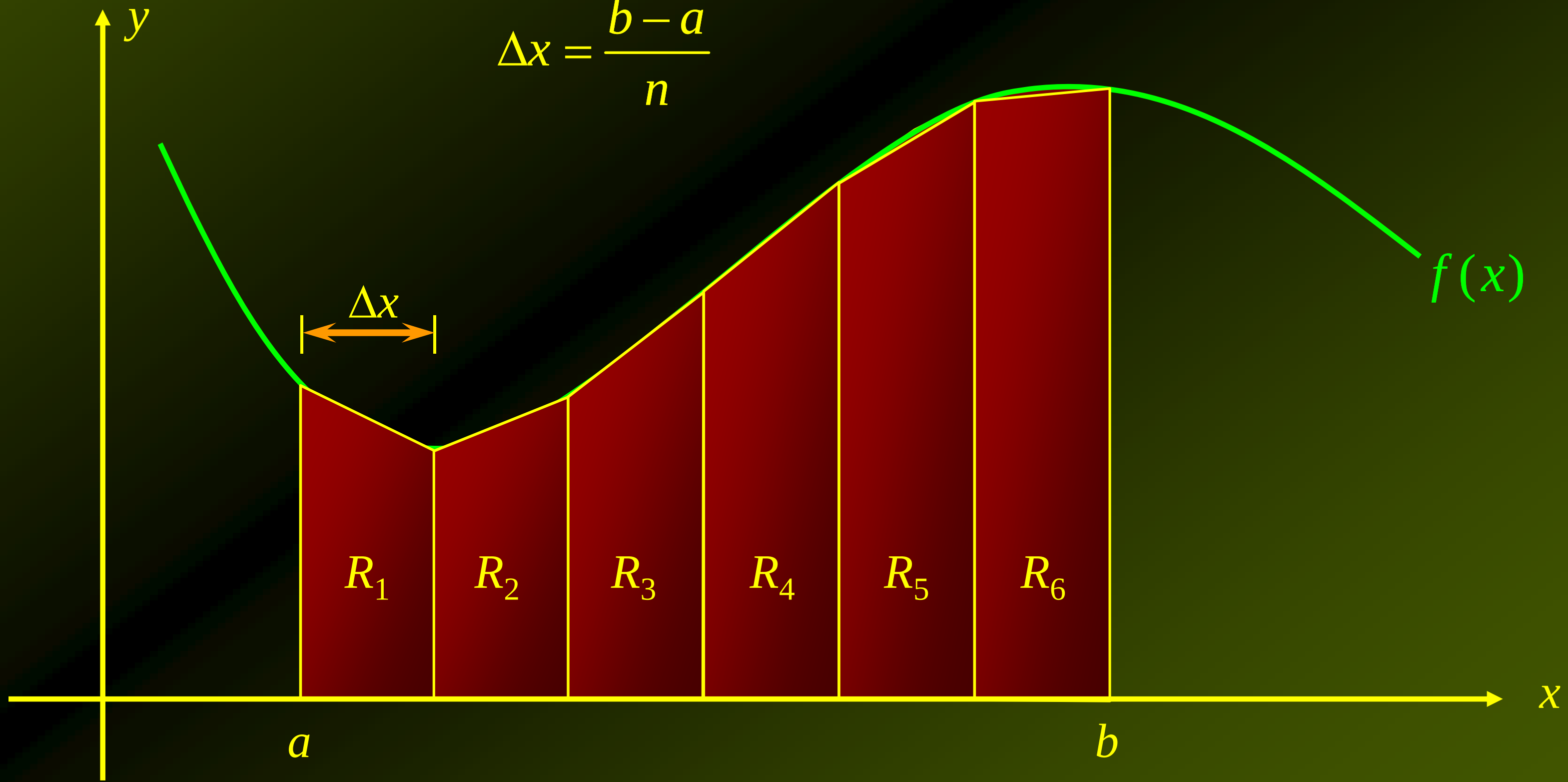
The trapezoidal rule is based on the notion of dividing the area to be evaluated into trapezoids that approximate the area under the curve:



The Trapezoidal Rule

The increments Δx used for each trapezoid are obtained by dividing the interval into n equal segments (in our example $n = 6$):

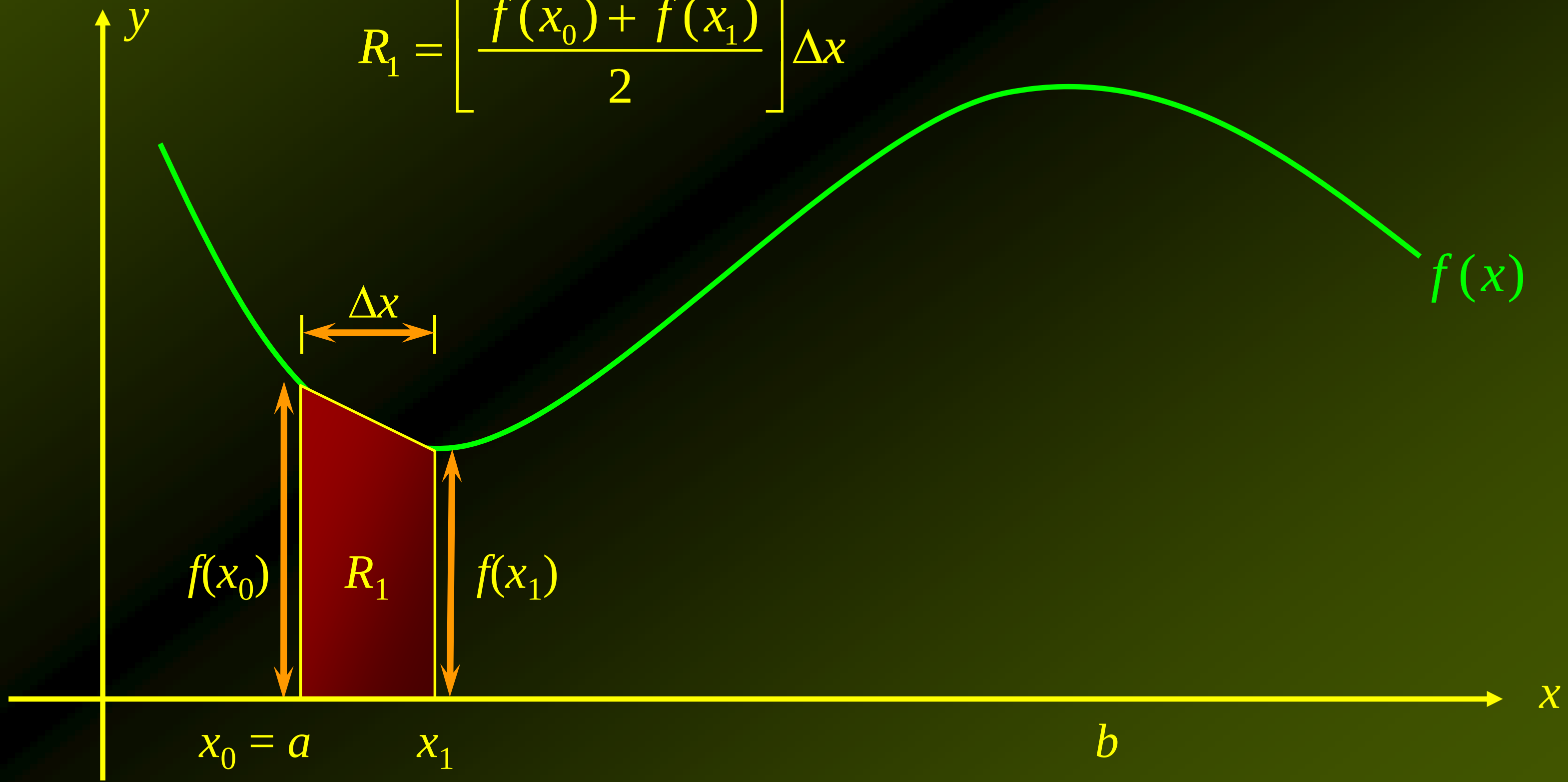
$$\Delta x = \frac{b-a}{n}$$



The Trapezoidal Rule

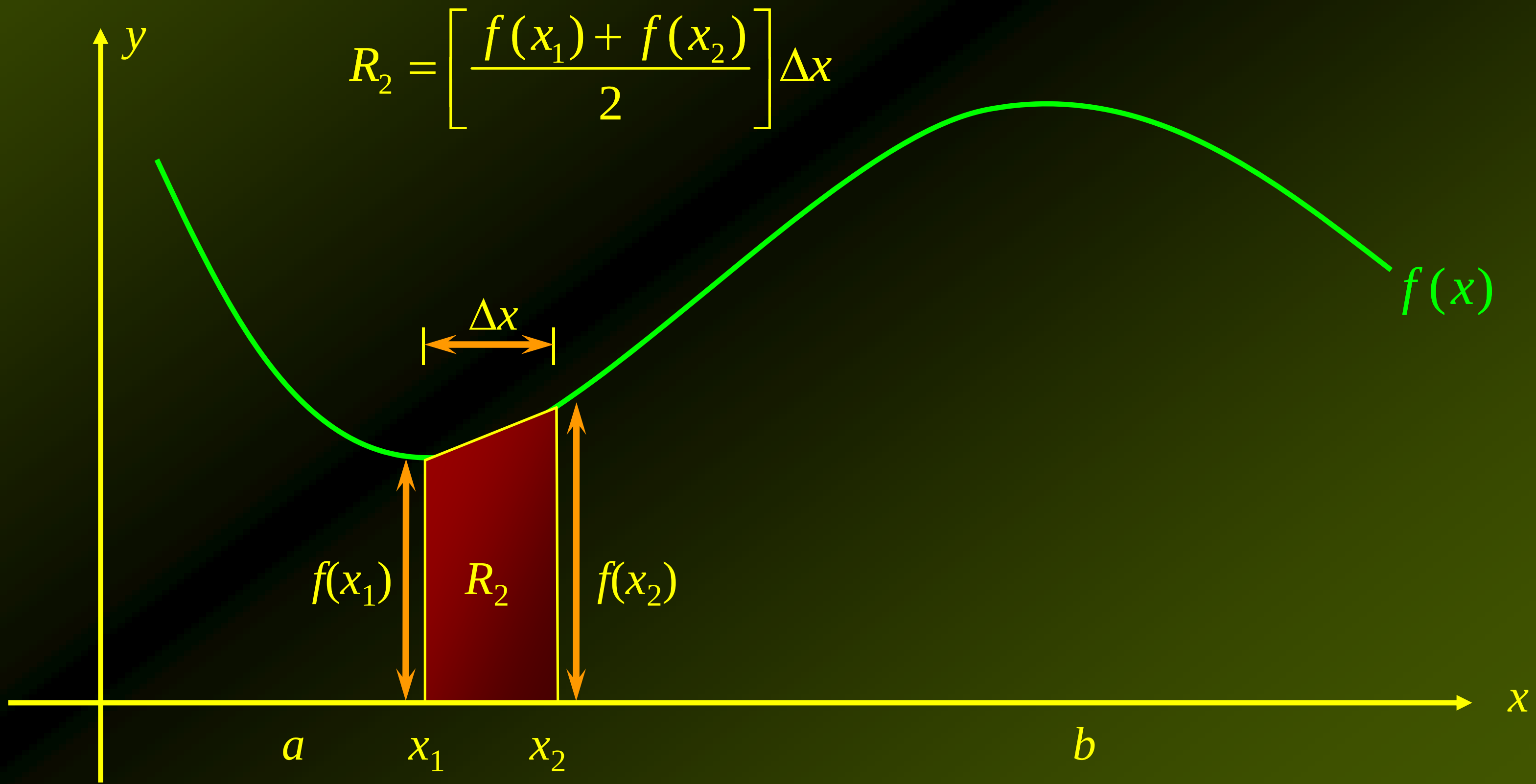
The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:

$$R_1 = \left[\frac{f(x_0) + f(x_1)}{2}\right]\Delta x$$



The Trapezoidal Rule

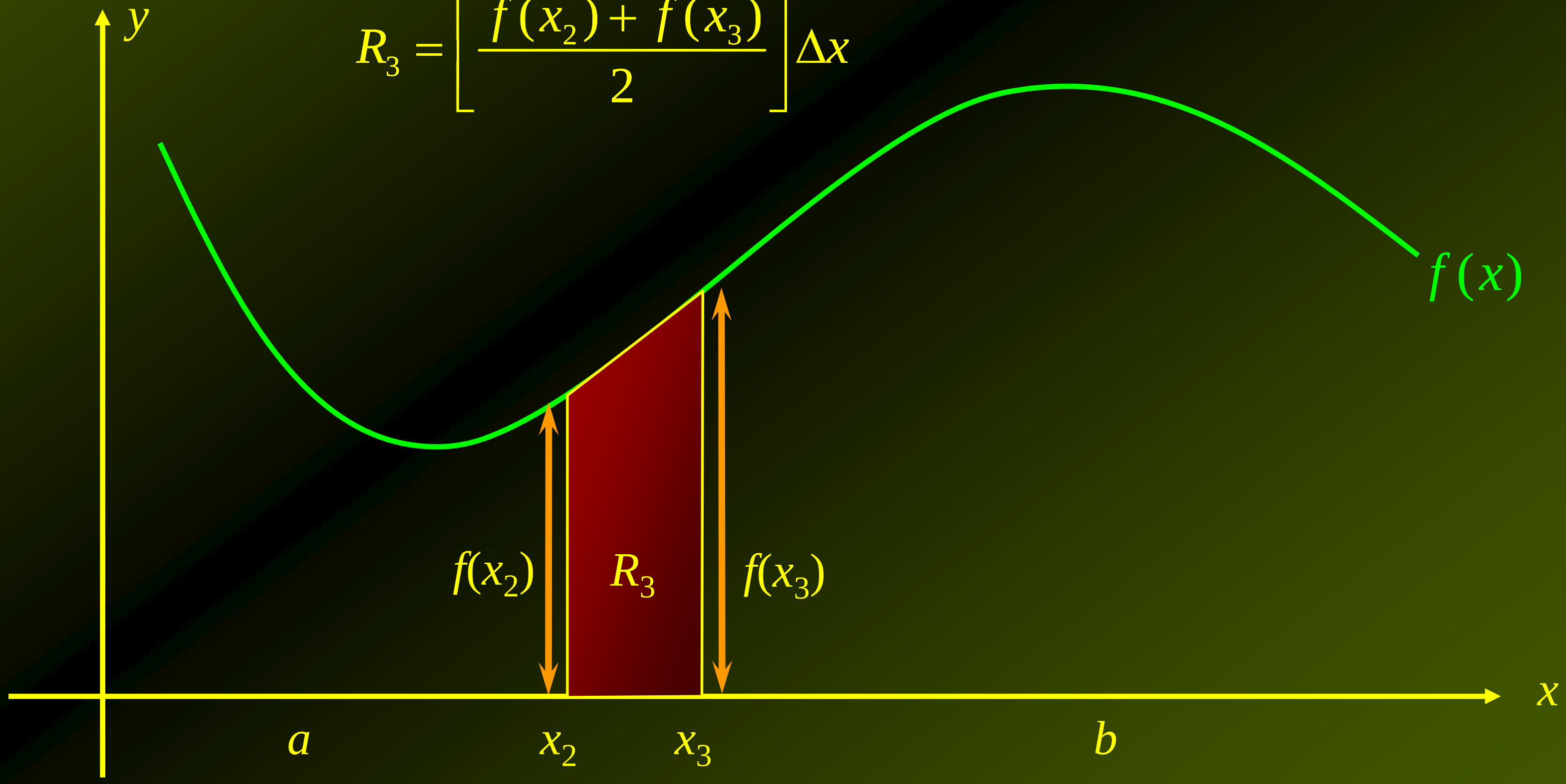
The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:



The Trapezoidal Rule

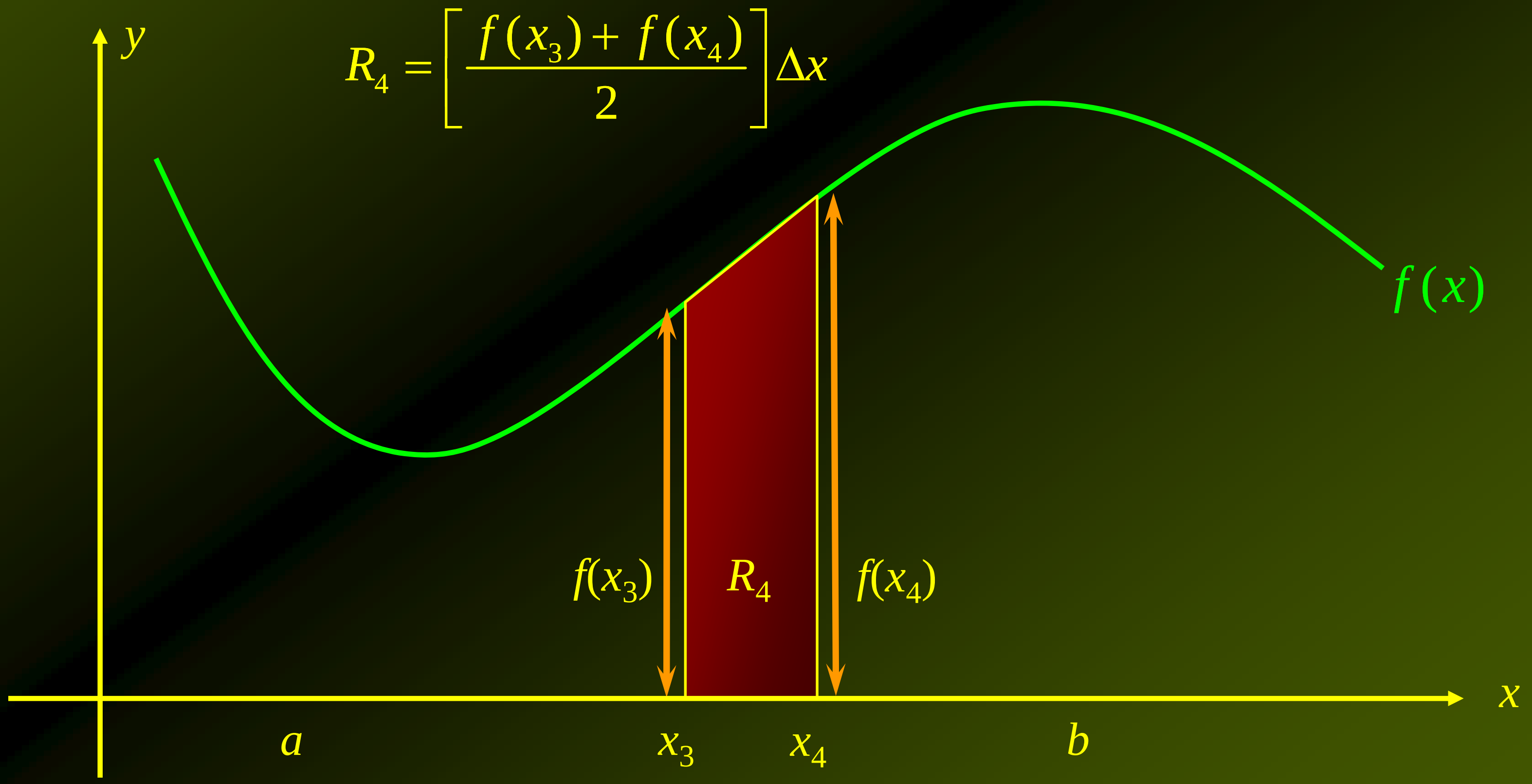
The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:

$$R_3 = \left[\frac{f(x_2) + f(x_3)}{2}\right]\Delta x$$



The Trapezoidal Rule

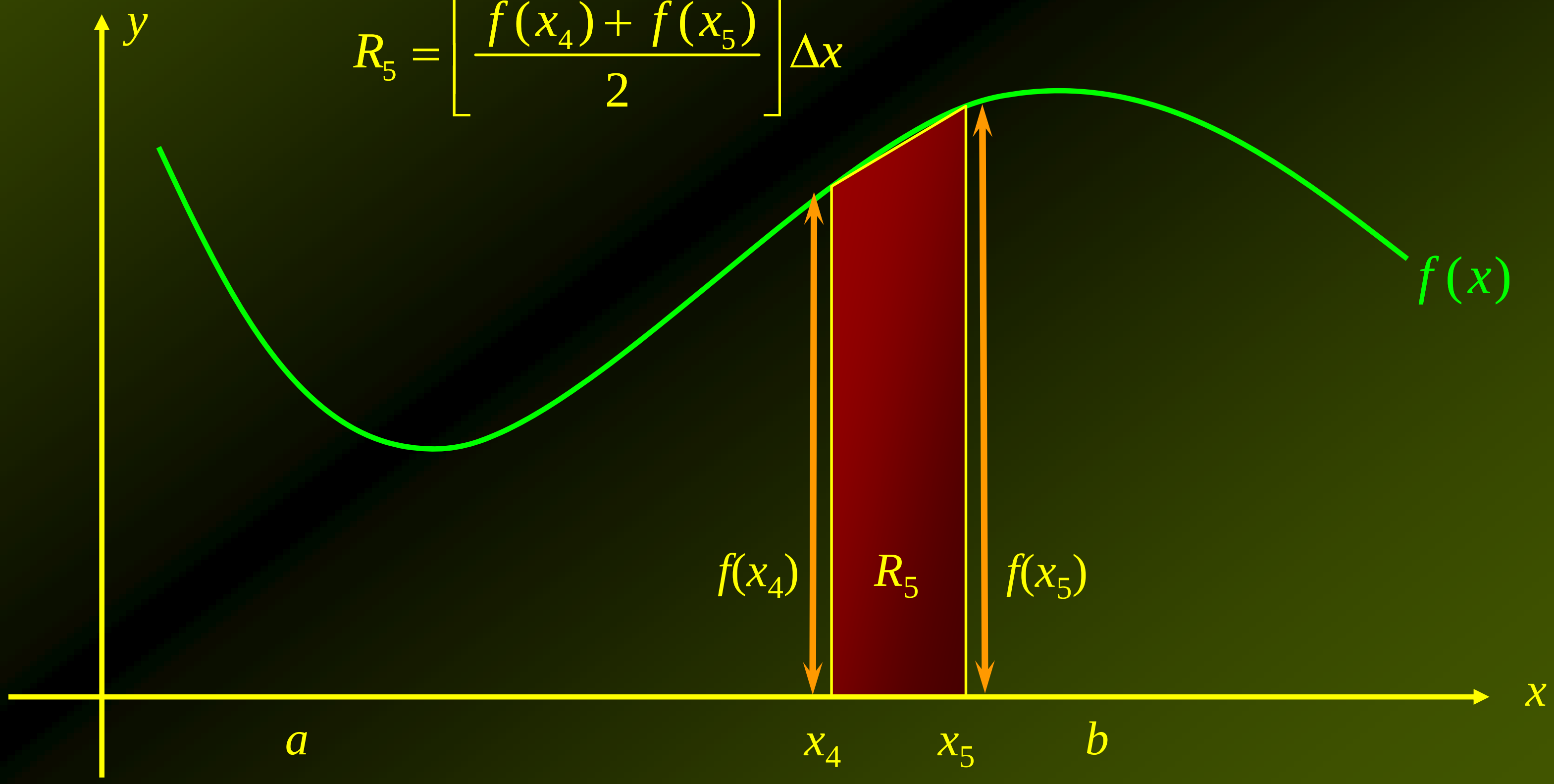
The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:



The Trapezoidal Rule

The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:

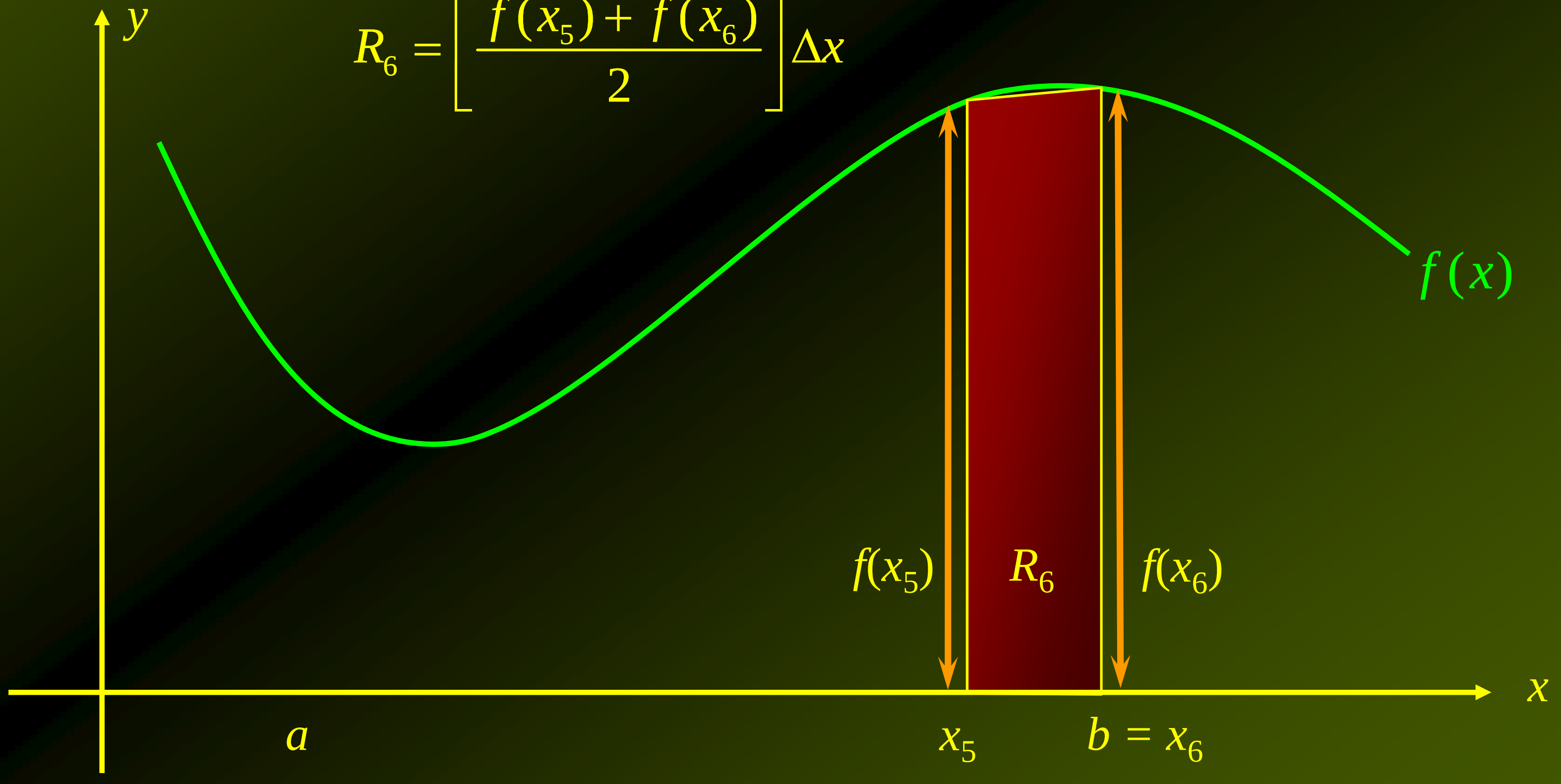
$$R_5 = \left[\frac{f(x_4) + f(x_5)}{2}\right]\Delta x$$



The Trapezoidal Rule

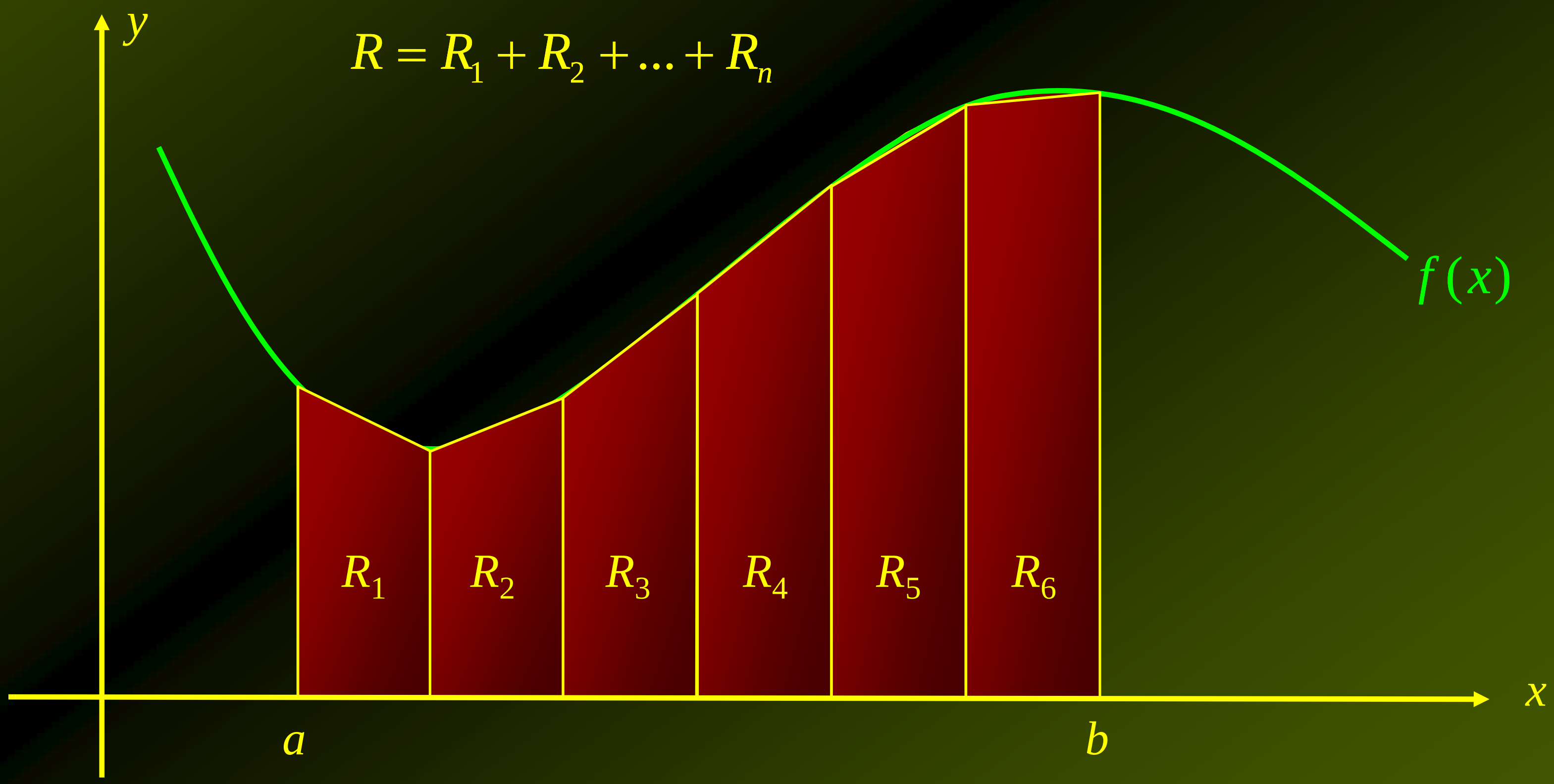
The area of each trapezoid is calculated by multiplying its base, Δx, by its average height:

$$R_6 = \left[\frac{f(x_5) + f(x_6)}{2}\right]\Delta x$$



The Trapezoidal Rule

Adding the areas R_1 through R_n ($n = 6$ in this case) of the trapezoids gives an approximation of the desired area of the region R:



The Trapezoidal Rule

Adding the areas R_1 through R_n of the trapezoids yields the following rule:

Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{2}[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}$.

Example 1

Approximate the value of $\int_1^2 \frac{1}{x}\,dx$ using the trapezoidal rule with $n = 10$.

Compare this result with the exact value of the integral.

Solution:

Here, $a = 1$, $b = 2$, and $n = 10$, so

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

and

$x_0 = 1$, $x_1 = 1.1$, $x_2 = 1.2$, $x_3 = 1.3$, … , $x_9 = 1.9$, $x_{10} = 1.10$.

Example 1– *Solution*

cont'd

The trapezoidal rule yields

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.1}{2}[1+2\left(\frac{1}{1.1}\right)+2\left(\frac{1}{1.2}\right)+2\left(\frac{1}{1.3}\right)...+2\left(\frac{1}{1.9}\right)+\frac{1}{2}] \approx 0.693771$$

By computing the actual value of the integral we get

$$\int_1^2 \frac{1}{x}\,dx = \ln x\Big|_1^2 = \ln 2 - \ln 1 = \ln 2 \approx 0.693147$$

Thus the trapezoidal rule with $n = 10$ yields a result with an error of 0.000624 to six decimal places.

Applied Example 2 – *Consumers' Surplus*

The demand function for a certain brand of perfume is given by

$$p = D(x) = \sqrt{10,000 - 0.01x^2}$$

where *p* is the unit price in dollars and *x* is the quantity demanded each week, measured in ounces.

Find the consumers' surplus if the market price is set at $60 per ounce.

Applied Example 2 – *Solution*

When $p = 60$, we have

$$\sqrt{10,000 - 0.01x^2} = 60$$

$$10,000 - 0.01x^2 = 3,600$$

$$x^2 = 640,000$$

or $x = 800$ since x must be nonnegative.

Next, using the consumers' surplus formula with $\bar{p} = 60$ and $\bar{x} = 800$, we see that the consumers' surplus is given by

$$CS = \int_0^{800} \sqrt{10,000 - 0.01x^2}\,dx - (60)(800)$$

Applied Example 2 – *Solution*

cont'd

It is not easy to evaluate this definite integral by finding an antiderivative of the integrand. But we can, instead, use the trapezoidal rule.

We can use the trapezoidal rule with $a = 0$, $b = 800$, and $n = 10$.

$$\Delta x = \frac{b-a}{n} = \frac{800-0}{10} = \frac{800}{10} = 80$$

and $x_0 = 0$, $x_1 = 80$, $x_2 = 160$, $x_3 = 240$, … , $x_9 = 720$, $x_{10} = 800$.

The trapezoidal rule yields

$$CS = \int_0^{800} \sqrt{10{,}000 - 0.01x^2}\,dx - (60)(800)$$

Applied Example 2 – *Solution*

cont'd

$$\approx \frac{80}{2}\left[100 + 2\sqrt{10,000 - (0.01)(80)^2} + 2\sqrt{10,000 - (0.01)(160)^2} + \ldots\right.$$

$$\left.\ldots + 2\sqrt{10,000 - (0.01)(720)^2} + \sqrt{10,000 - (0.01)(800)^2}\right]$$

The trapezoidal rule yields

$$CS = \int_0^{800} \sqrt{10,000 - 0.01x^2}\,dx - (60)(800)$$

$$= 40(100 + 199.3590 + 197.4234 + 194.1546 + 189.4835$$

$$+ 183.3030 + 175.4537 + 165.6985$$

$$+ 153.6750 + 138.7948 + 60)$$

$$\approx 70,293.82$$

Applied Example 2 – *Solution*

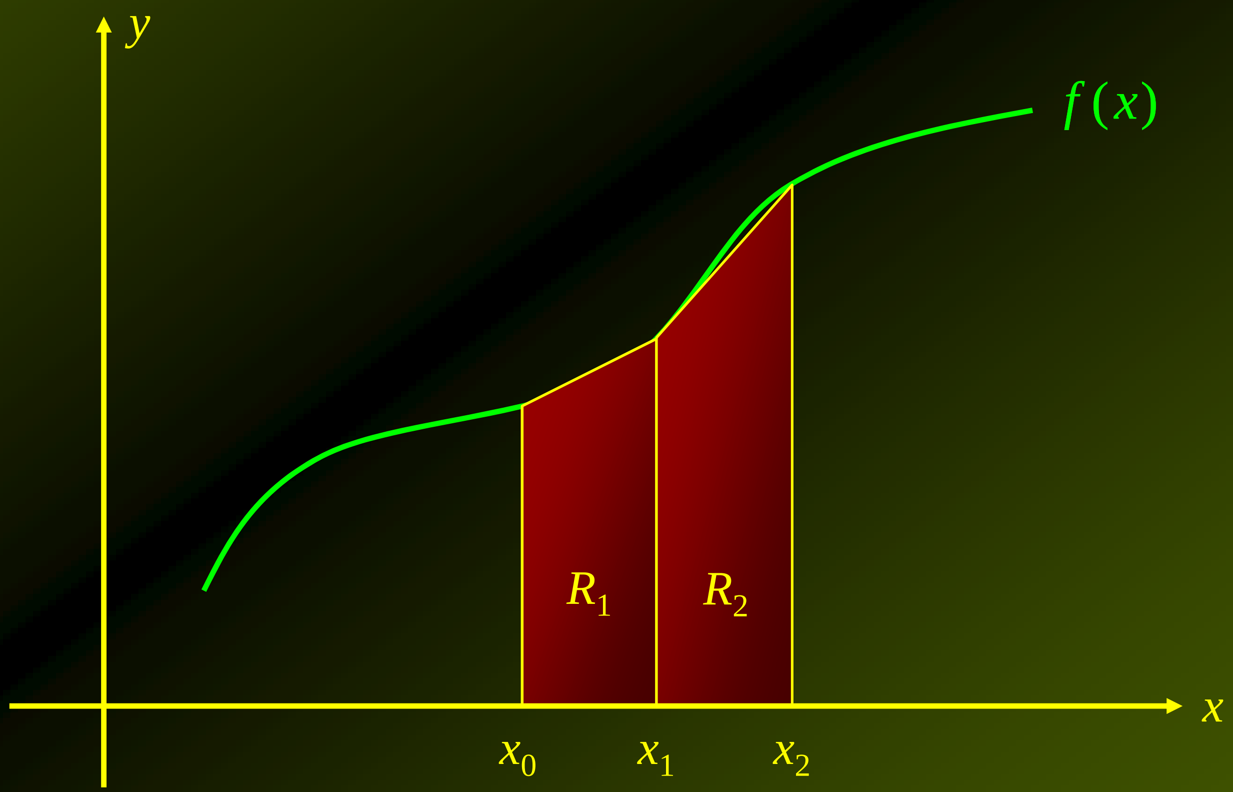
cont'd

Therefore, the consumers' surplus is approximately

$$70,294 - 48,000, \text{ or } \$22,294$$

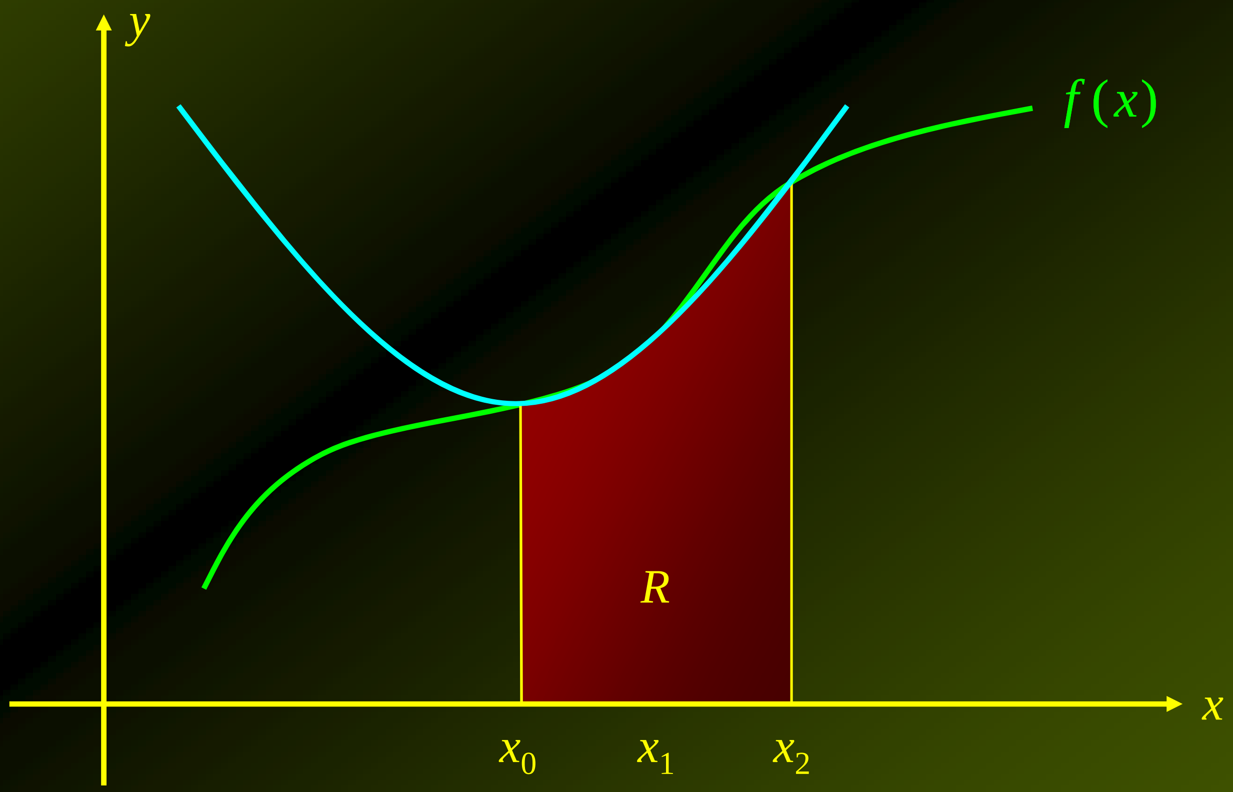
Simpson's Rule

We've seen that the trapezoidal rule approximates the area under the curve by adding the areas of trapezoids under the curve:



Simpson's Rule

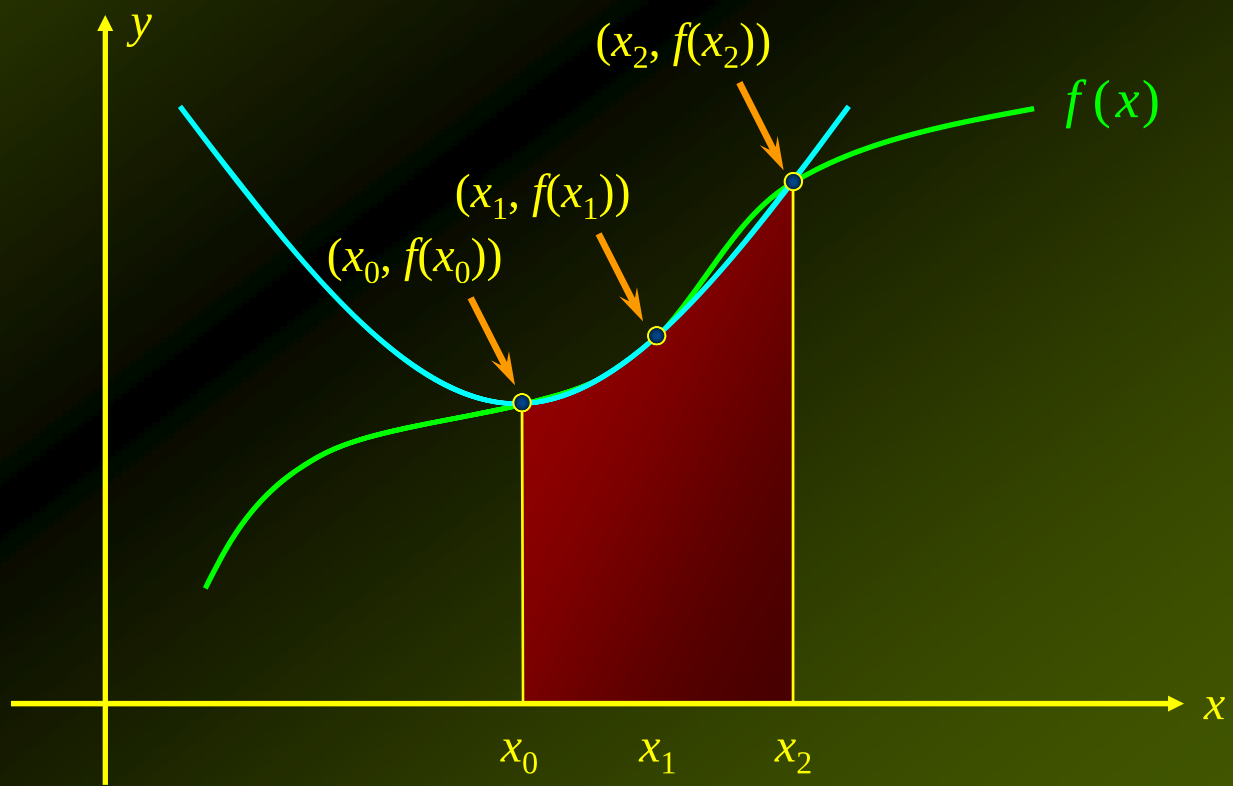
The Simpson's rule improves upon the trapezoidal rule by approximating the area under the curve by the area under a parabola, rather than a straight line:



Simpson's Rule

Given any three nonlinear points there is a unique parabola that passes through the given points.

We can approximate the function $f(x)$ on $[x_0, x_2]$ with a quadratic function whose graph contain these three points:



Simpson's Rule

Simpson's rule approximates the area under the curve of a function *f*(*x*) using a quadratic function:

Simpson's rule

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+2f(x_4)$$
$$+\cdots+4f(x_{n-1})+f(x_n)]$$

where $\Delta x = \frac{b-a}{n}$ **and** n **is even.**

Example 3

Find an approximation of $\int_1^2 \frac{1}{x}\,dx$ using Simpson's rule with $n = 10$.

Solution:

Here, $a = 1$, $b = 2$, and $n = 10$, so

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{10} = \frac{1}{10} = 0.1$$

Simpson's rule yields

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.1}{3}[f(1) + 4f(1.1) + 2f(1.2) + 4f(1.3) + 2f(1.4) + \cdots + 4f(1.9) + f(2)]$$

Example 3 – *Solution*

cont'd

$$= \frac{0.1}{3}\left[1 + 4\left(\frac{1}{1.1}\right) + 2\left(\frac{1}{1.2}\right) + 4\left(\frac{1}{1.3}\right) + 2\left(\frac{1}{1.4}\right) + \cdots + 4\left(\frac{1}{1.9}\right) + \frac{1}{2}\right]$$

$$\approx 0.693150$$

Recall that the trapezoidal rule with $n = 10$ yielded an approximation of 0.693771, with an error of 0.000624 from the value of $\ln 2 \approx 0.693147$ to six decimal places.

Simpson's rule yields an approximation with an error of 0.000003 to six decimal places, a definite improvement over the trapezoidal rule.

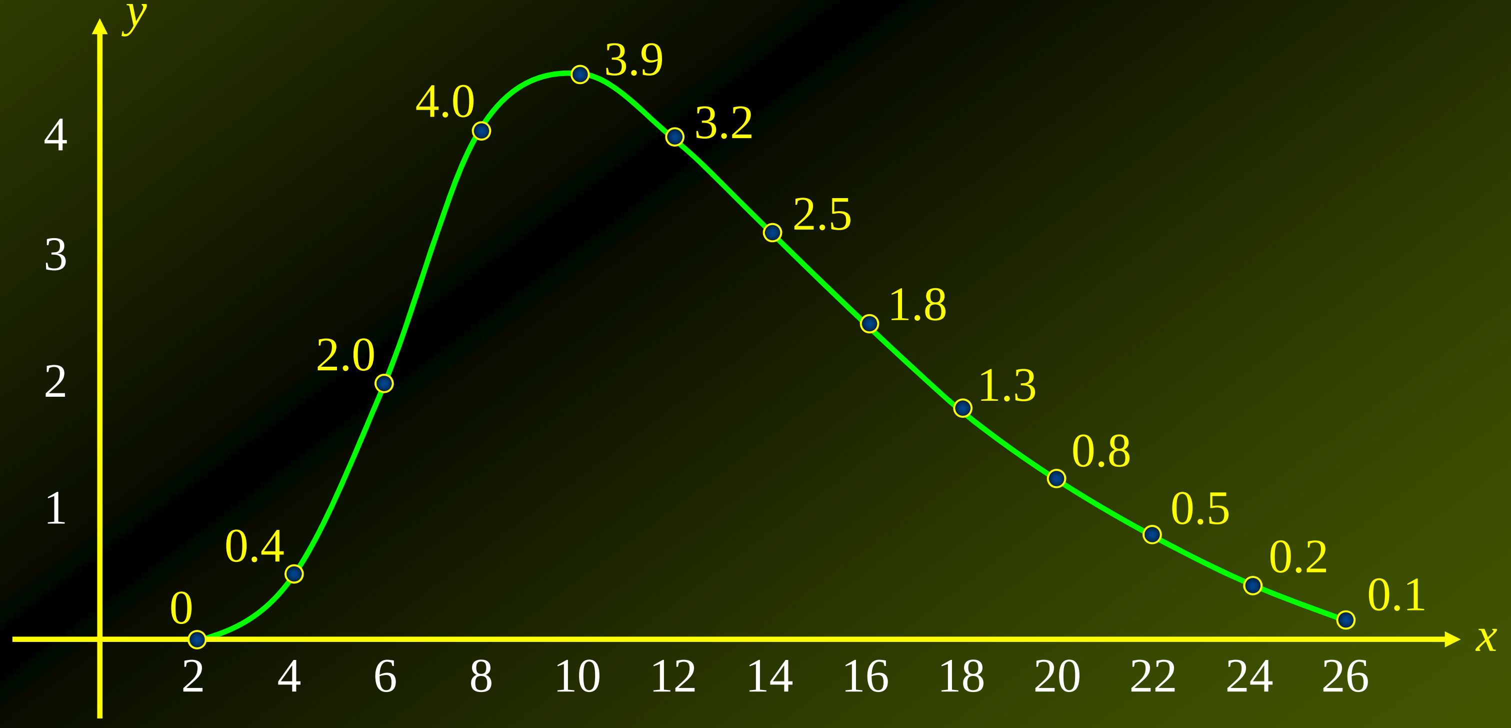
Applied Example 4 – *Cardiac Output*

One method of measuring cardiac output is to inject 5 to 10 mg of a dye into a vein leading to the heart.

After making its way through the lungs, the dye returns to the heart and is pumped into the aorta, where its concentration is measured at equal time intervals.

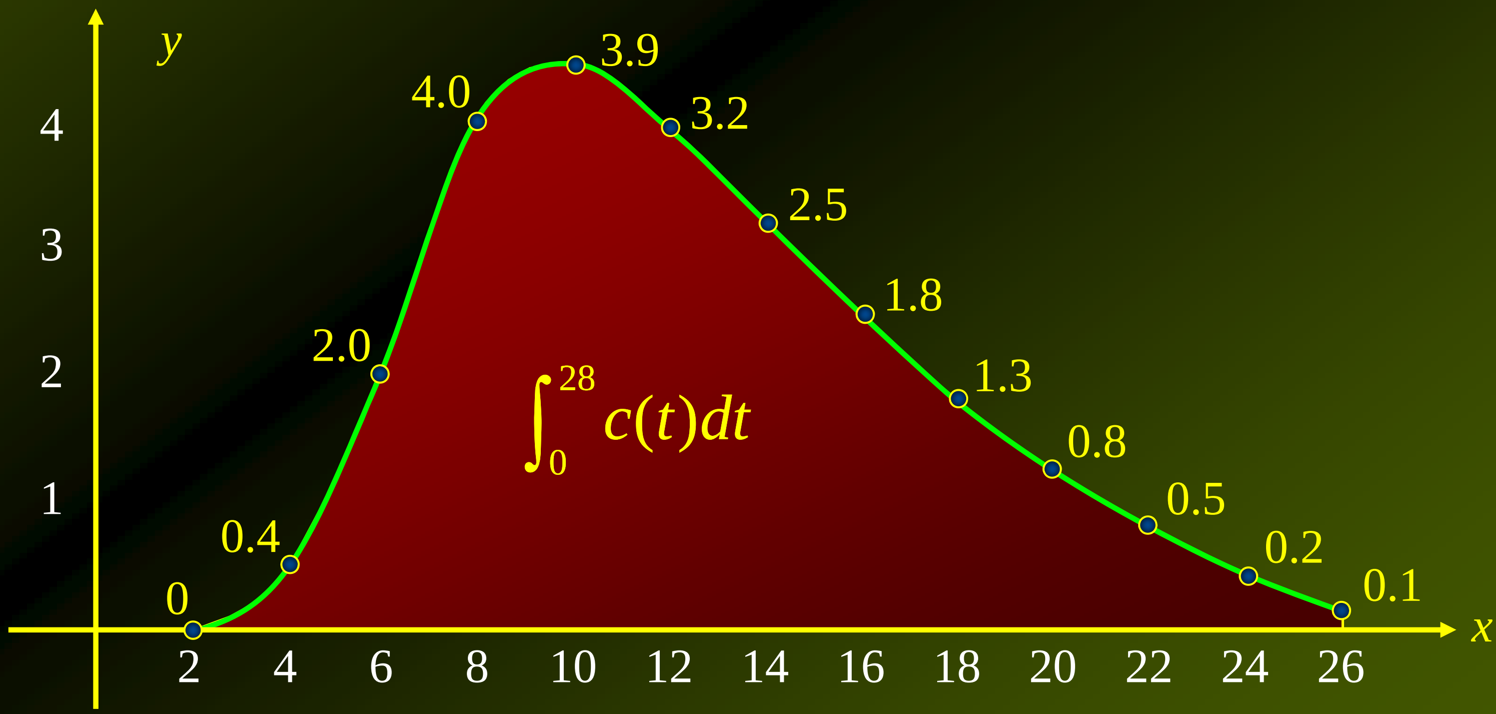
Applied Example 4 – *Cardiac Output* cont'd

The graph of $c(t)$ shows the concentration of dye in a person's aorta, measured in 2-second intervals after 5 mg of dye have been injected:



Applied Example 4 – *Cardiac Output* cont'd

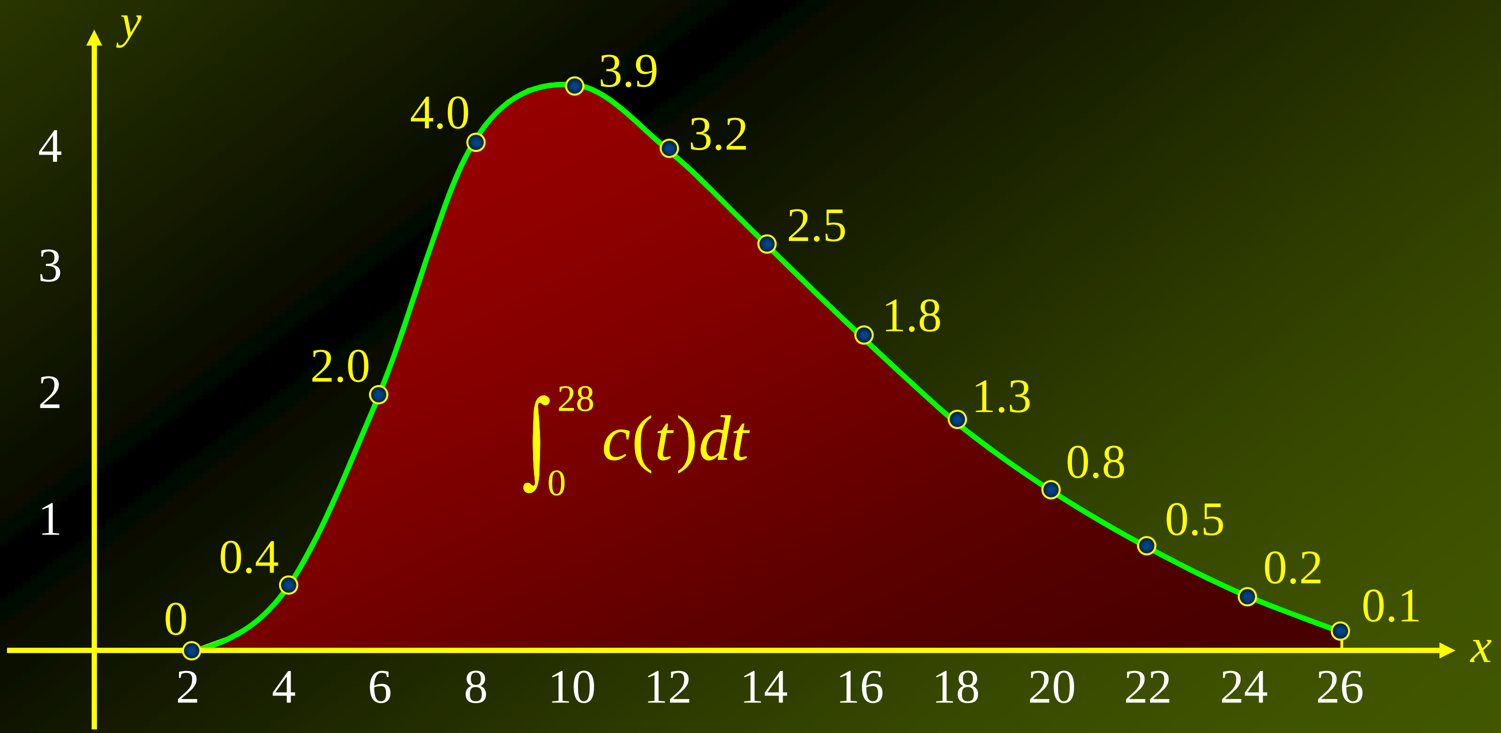
The person's cardiac output, measured in liters per minute (L/min) is computed using the formula $R = \dfrac{60D}{\int_0^{28} c(t)\,dt}$ where D is the quantity of dye injected.



Applied Example 4 – *Cardiac Output* cont'd

Use Simpson's rule with $n = 14$ to evaluate the integral and determine the person's cardiac output.

$$R = \frac{60D}{\int_0^{28} c(t)\,dt}$$



Applied Example 4 – *Solution*

We have $a = 0$, $b = 28$, $n = 14$, and $\Delta t = 2$, so that

$t_0 = 0$, $t_1 = 2$, $t_2 = 4$, $t_3 = 6$, … , $t_{14} = 28$.

Simpson's rule yields

$$\int_0^{28} c(t)\,dt \approx \frac{2}{3}[c(0) + 4c(2) + 2c(4) + 4c(6) + \ldots + 4c(26) + c(28)]$$

$$\approx \frac{2}{3}[0 + 4(0) + 2(0.4) + 4(2.0) + 2(4.0)$$

$$+ 4(4.4) + 2(3.9) + 4(3.2) + 2(2.5) + 4(1.8)$$

$$+ 2(1.3) + 4(0.8) + 2(0.5) + 4(0.2) + 0.1]$$

$$\approx 49.9$$

Applied Example 4 – *Solution*

cont'd

Therefore, the person's cardiac output is

$$R = \frac{60D}{\int_0^{28} c(t)\,dt}$$

$$\approx \frac{60(5)}{49.9}$$

$$\approx 6.0$$

or approximately 6.0 L/min.