7

ADDITIONAL TOPICS IN INTEGRATION

7.4 Improper Integrals

Improper Integrals

In many applications we are concerned with integrals that have unbounded intervals of integration.

These are called improper integrals.

We will now discuss problems that involve improper integrals.

Improper Integral of *f* over [*a*, ∞)

Let *f* be a continuous function on the unbounded interval [*a*, ∞). Then the improper integral of *f* over [*a*, ∞) is defined by

$$\int_a^\infty f(x)dx = \lim_{b\to\infty}\int_a^b f(x)dx$$

if the limit exists.

Example 2

Evaluate $\int_2^\infty \frac{1}{x}\,dx$ if it converges.

Solution:

$$\int_2^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_2^b \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \ln x\Big|_2^b$$

$$= \lim_{b\to\infty}(\ln b - \ln 2)$$

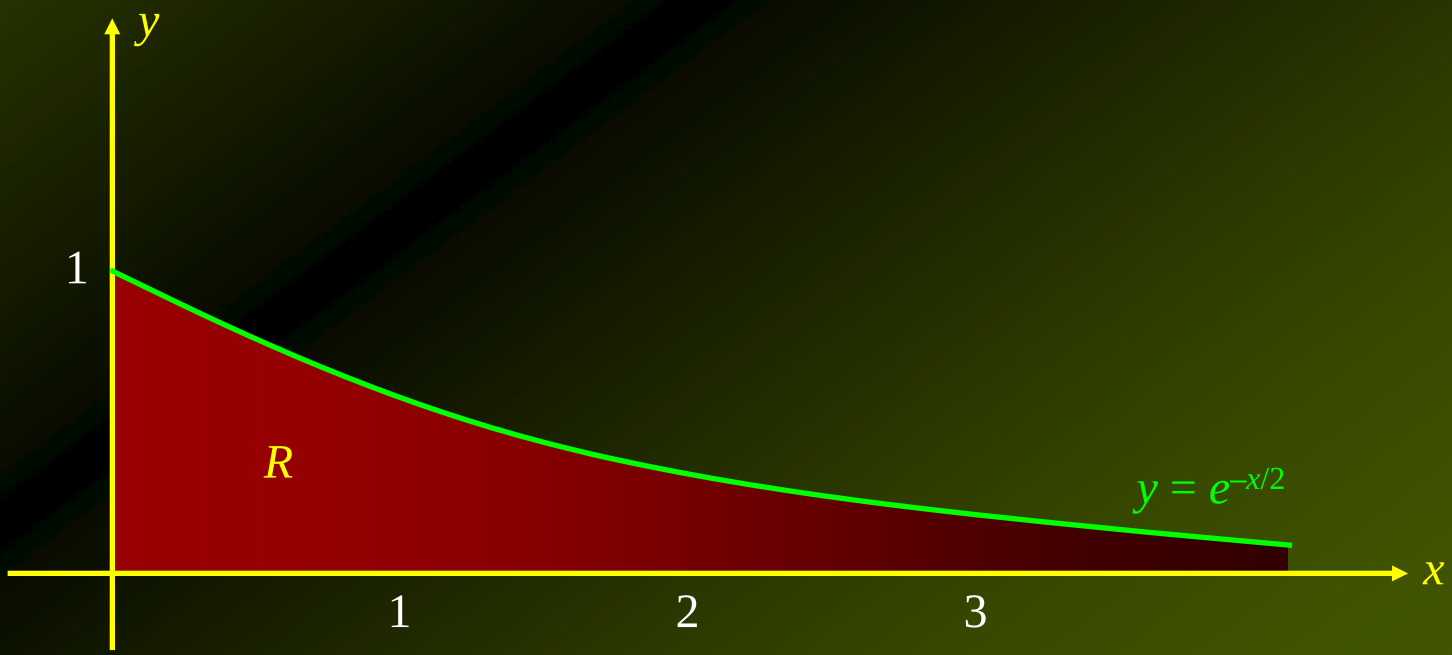
Since $\ln b \to \infty$, as $b \to \infty$ the limit does not exist, and we conclude that the given improper integral is divergent.

Example 3

Find the area of the region R under the curve $y = e^{-x/2}$ for $x \geq 0$.

Solution:

The required area is shown in the diagram below:



Example 3 – *Solution*

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Taking $b > 0$, we compute the area of the region under the curve $y = e^{-x/2}$ from $x = 0$ to $x = b$,

$$I(b) = \int_0^b e^{-x/2}\,dx = -2e^{-x/2}\Big|_0^b = -2e^{-b/2} + 2$$

Then, the area of the region R is given by

$$I(b) = \lim_{b\to\infty}(2 - 2e^{-b/2}) = 2 - 2\lim_{b\to\infty}\frac{1}{e^{b/2}} = 2$$

or 2 square units.

Improper Integral of f over $(-\infty, b]$

Let f be a continuous function on the unbounded interval $(-\infty, b]$. Then the improper integral of f over $(-\infty, b]$ is defined by

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$$

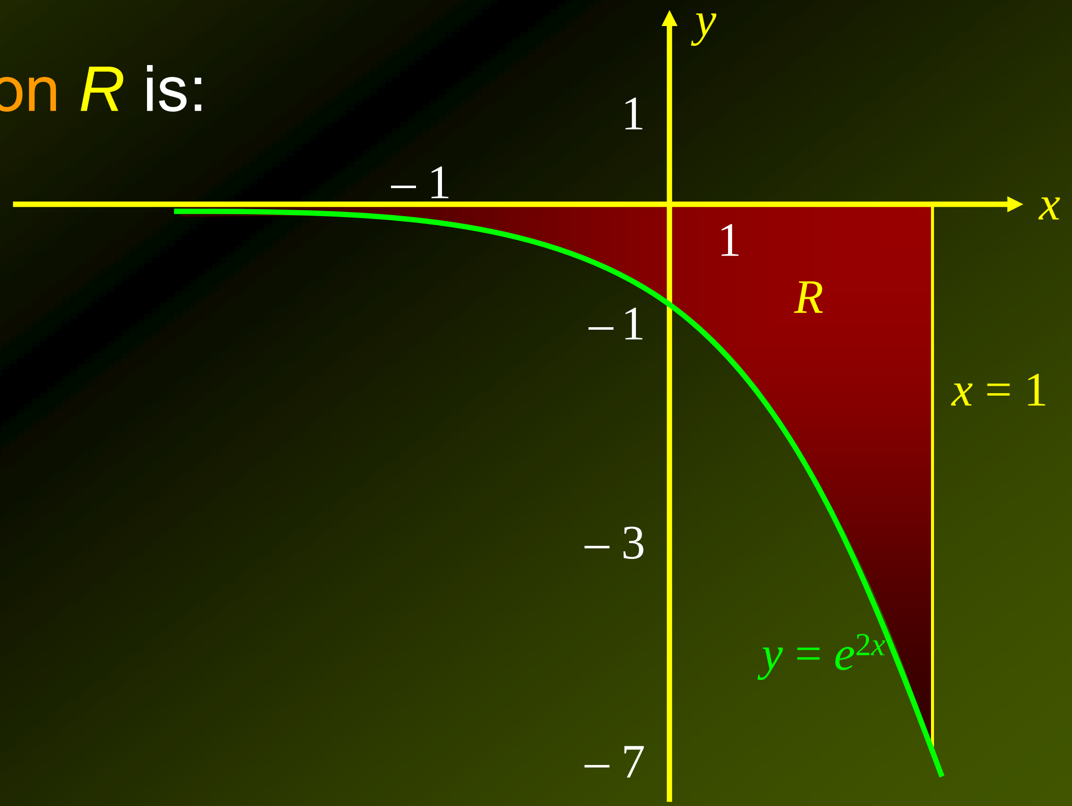
if the limit exists.

Example 4

Find the area of the region R bounded above by the x-axis, below by $y = -e^{2x}$, and on the right, by the line $x = 1$.

Solution:

The graph of region R is:



Example 4 – *Solution*

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Taking $a < 1$, compute

$$I(a) = \int_a^1 [0 - (-e^{2x})]dx = \int_a^1 e^{2x} dx = \frac{1}{2}e^{2x}\Big|_a^1 = \frac{1}{2}e^2 - \frac{1}{2}e^{2a}$$

Then, the area under the required region *R* is given by

$$\lim_{a \to -\infty} I(a) = \lim_{a \to -\infty} \left(\frac{1}{2}e^2 - \frac{1}{2}e^{2a} \right)$$

$$= \frac{1}{2}e^2 - \lim_{a \to -\infty} \frac{1}{2}e^{2a}$$

$$= \frac{1}{2}e^2$$

Improper Integral Unbounded on Both Sides

Improper Integral of f over $(-\infty, \infty)$

Let f be a continuous function over the unbounded interval $(-\infty, \infty)$.

Let c be any real number and suppose both the improper integrals

$$\int_{-\infty}^{c} f(x)dx \quad \text{and} \quad \int_{c}^{\infty} f(x)dx$$

are convergent.

Then, the improper integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx$$

Example 5

Evaluate the improper integral $\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ and give a geometric interpretation of the result.

Solution:

Take the number *c* to be zero and evaluate first for the interval (–∞, 0):

$$\int_{-\infty}^{0} xe^{-x^2}\,dx = \lim_{a\to-\infty}\int_{a}^{0} xe^{-x^2}\,dx$$

$$= \lim_{a\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_{a}^{0}$$

$$= \lim_{a\to-\infty}\left(-\frac{1}{2}+\frac{1}{2}e^{-a^2}\right) = -\frac{1}{2}$$

Example 5 – *Solution*

cont'd

Now evaluate for the interval $(0, \infty)$:

$$\int_0^{\infty} xe^{-x^2}\,dx = \lim_{b\to\infty}\int_0^b xe^{-x^2}\,dx$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-x^2}\right)\Bigg|_0^b$$

$$= \lim_{b\to\infty}\left(-\frac{1}{2}e^{-b^2} + \frac{1}{2}\right) = \frac{1}{2}$$

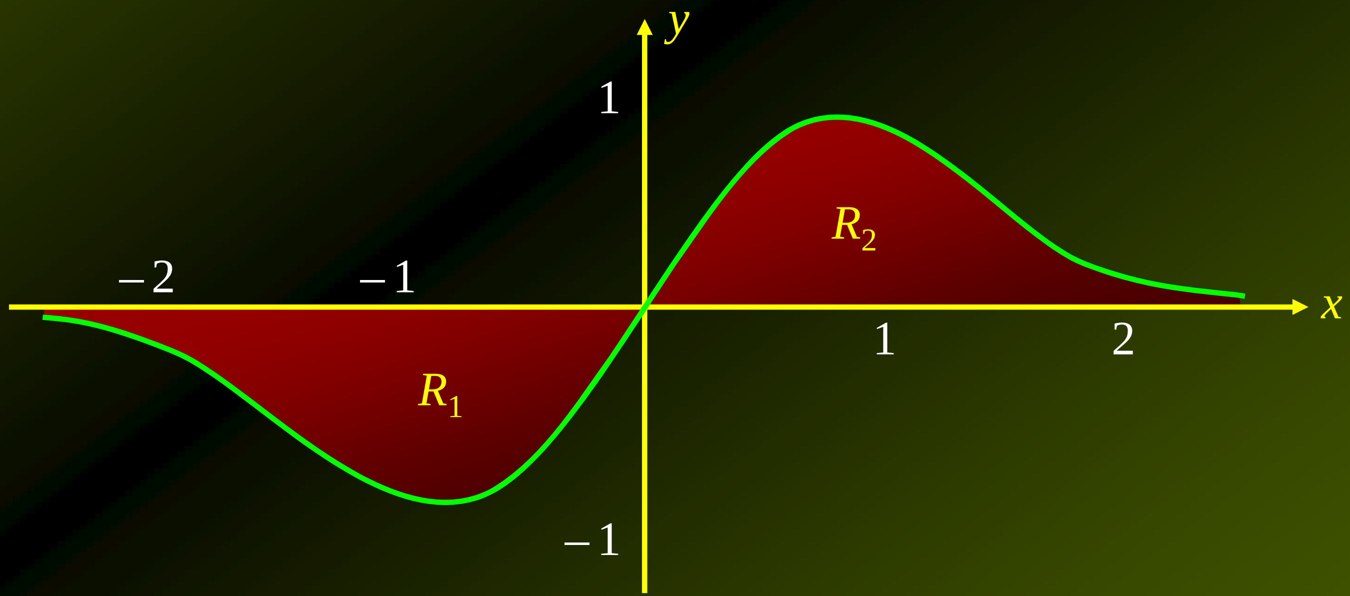
Therefore,

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx = \int_{-\infty}^{0} xe^{-x^2}\,dx + \int_0^{\infty} xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Example 5 – *Solution*

cont'd

Below is the graph of $y = xe^{-x^2}$, showing the regions of interest R_1 and R_2:



Example 5 – *Solution*

cont'd

Region R_1 lies below the x-axis, so its area is negative ($R_1 = -½$).

While the symmetrically identical region R_2 lies above the x-axis, so its area is positive ($R_2 = ½$).

Thus, adding the areas of the two regions yields zero:

$$R = R_1 + R_2 = -\frac{1}{2} + \frac{1}{2} = 0$$