7 ADDITIONAL TOPICS IN INTEGRATION

7.5 Volumes of Solids of Revolution

Finding the Volume of a Solid of Revolution

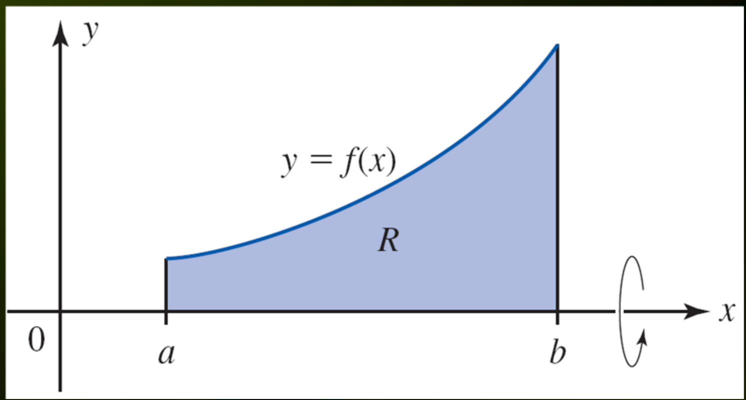
Finding the Volume of a Solid of Revolution

In this section we use a Riemann sum to find a formula for computing the volume of a solid that results when a region is revolved about an axis.

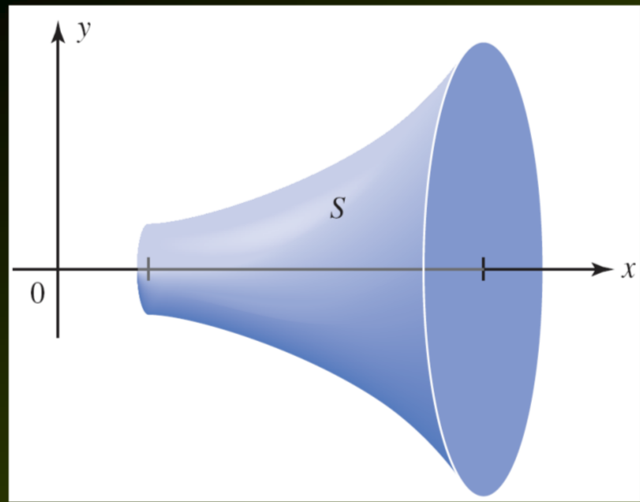
The solid is called a solid of revolution.

Finding the Volume of a Solid of Revolution

To find the volume of a solid of revolution, suppose the plane region under the curve defined by a nonnegative continuous function $y = f(x)$ between $x = a$ and $x = b$ is revolved about the x-axis (Figure 15).



(a) Region *R* under the curve



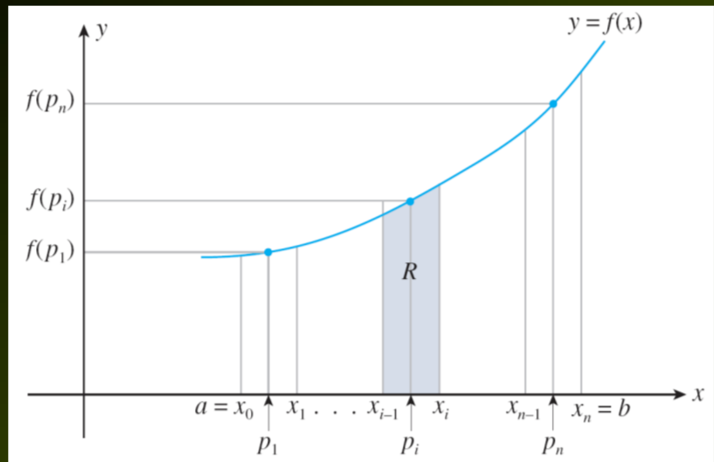
(b) Solid obtained by revolving *R* about the *x*-axis

Figure 15

Finding the Volume of a Solid of Revolution

To derive a formula for finding the volume of the solid of revolution, we divide the interval [a, b] into n subintervals, each of equal length, by means of the points $x_0 = a, x_1, x_2, \ldots, x_n = b$, so that the length of each subinterval is given by $\Delta x = (b - a)/n$.

Also, let $p_1, p_2, \ldots, p_n$ be points in the subintervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]$, respectively (Figure 16).



The region R is revolved about the x-axis.

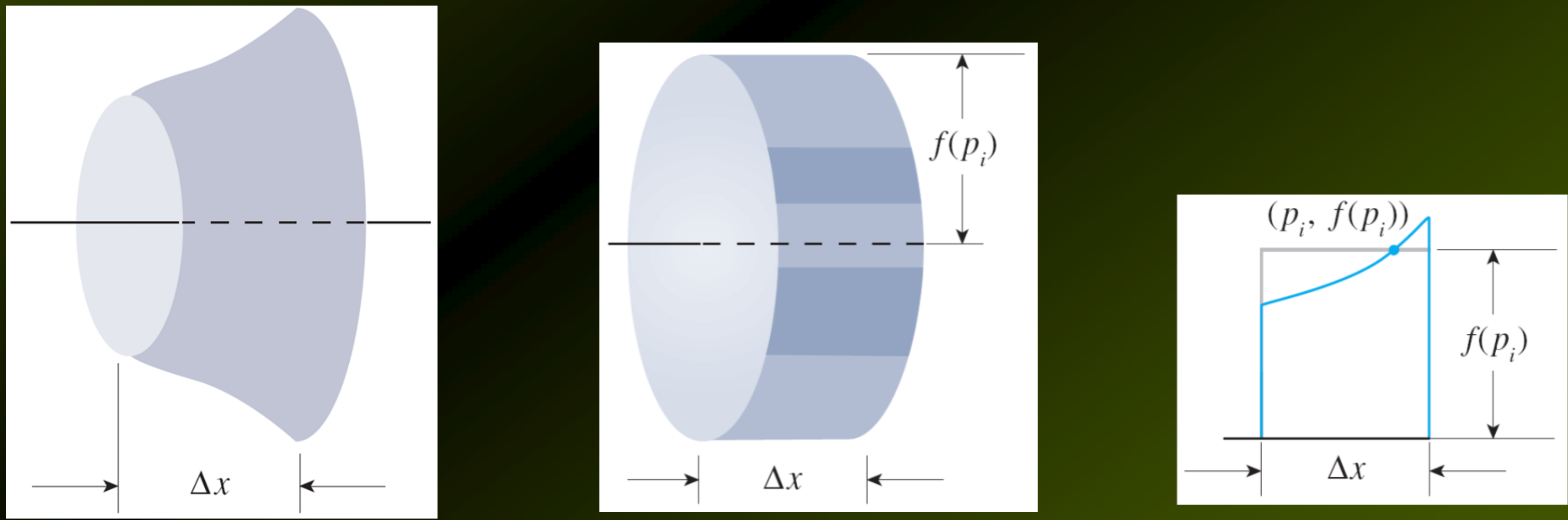
Figure 16

Finding the Volume of a Solid of Revolution

Let's concentrate on that part of the solid of revolution swept out by revolving the region under the curve between $x = x_{i-1}$ and $x = x_i$ about the x-axis.

Finding the Volume of a Solid of Revolution

The volume ΔV_i of this object, shown in Figure 17a, may be approximated by the volume of the disk shown in Figure 17b, of radius $f(p_i)$ and width Δx, obtained by revolving the rectangular region of height $f(p_i)$ and width Δx about the x-axis (Figure 17c).



(a) Volume ΔV_i of object

(b) Volume of disk

(c) Approximating rectangle

Figure 17

Finding the Volume of a Solid of Revolution

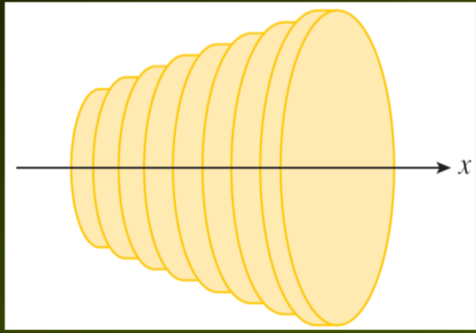
Since the volume of a disk (cylinder) is equal to $\pi r^2 h$, we see that

$$\Delta V_i \approx \pi[f(p_i)]^2\,\Delta x$$

This analysis suggests that the volume of the solid of revolution may be approximated by the sum of the volumes of n suitable disks—namely,

$$\pi[f(p_1)]^2\Delta x + \pi[f(p_2)]^2\Delta x + \cdots + \pi[f(p_n)]^2\Delta x$$

(shown in Figure 18).



The solid of revolution is approximated by n disks.

Figure 18

Finding the Volume of a Solid of Revolution

This last expression is a Riemann sum of the function $g(x) = \pi[f(x)]^2$ over the interval $[a, b]$. Letting n approach infinity, we obtain the following formula.

Volume of a Solid of Revolution

The volume V of the solid of revolution obtained by revolving the region below the graph of a nonnegative function $y = f(x)$ from $x = a$ to $x = b$ about the x-axis is

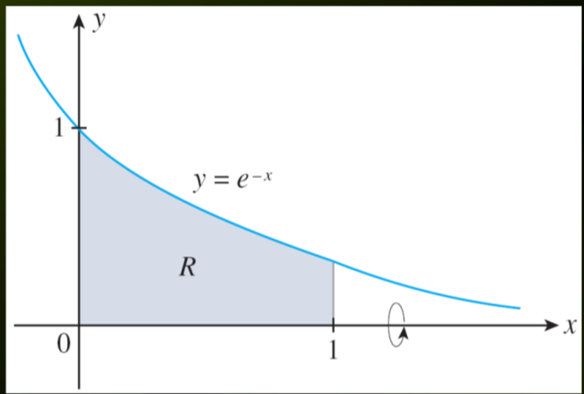
$$V = \pi \int_a^b [f(x)]^2 \, dx \qquad \textbf{(17)}$$

Example 1

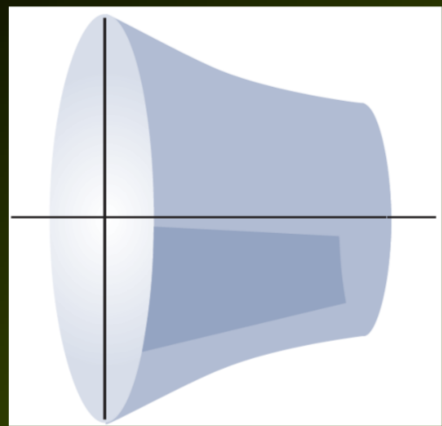
Find the volume of the solid of revolution obtained by revolving the region under the curve $y = f(x) = e^{-x}$ from $x = 0$ to $x = 1$ about the x-axis.

Solution:

The region under the curve and the resulting solid of revolution are shown in Figure 19.



(a) The region under the curve $y = e^{-x}$ from $x = 0$ to $x = 1$



(b) The solid of revolution obtained when R is revolved about the x-axis

Figure 19

Example 1 – *Solution*

cont'd

Using Formula (17), we find that the required volume is given by

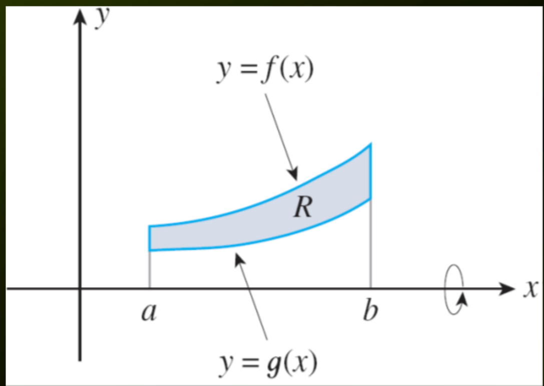
$$\pi\int_0^1 (e^{-x})^2\, dx = \pi\int_0^1 e^{-2x}\, dx$$

$$= -\frac{\pi}{2} e^{-2x} \Big|_0^1$$

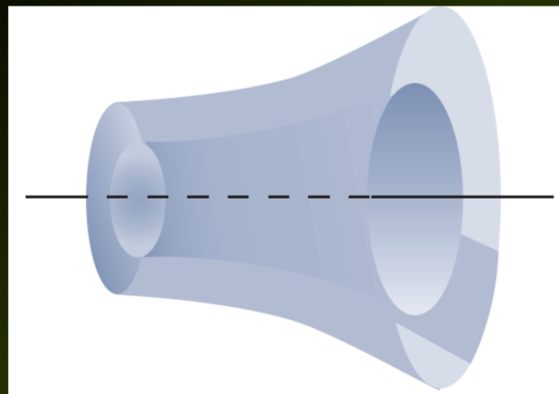
$$= \frac{\pi}{2}(1 - e^{-2})$$

Finding the Volume of a Solid of Revolution

Next, let's consider the solid of revolution obtained by revolving the region *R* bounded above by the graph of the nonnegative function $f(x)$ and below by the graph of the nonnegative function $g(x)$, from $x = a$ to $x = b$, about the x-axis (Figure 21).



(a) *R* is the region bounded by the curves $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$.



(b) The solid of revolution obtained by revolving *R* about the x-axis.

Figure 21

Finding the Volume of a Solid of Revolution

To derive a formula for computing the volume of this solid of revolution, observe that the required volume is the volume of the solid of revolution obtained by revolving the region under the curve $y = f(x)$ from $x = a$ to $x = b$ about the x-axis *minus* the volume of the solid of revolution obtained by revolving the region under the curve $y = g(x)$ from $x = a$ to $x = b$ about the x-axis.

Thus, the required volume is given by

$$V = \pi\int_a^b [f(x)]^2\, dx \;?\; \pi\int_a^b [g\ x)]^2\, dx \qquad \textbf{(18)}$$

or

$$V = \pi\int_a^b \left\{[f(x)]^2 \;?\; [g\ x)]^2\right\} dx$$