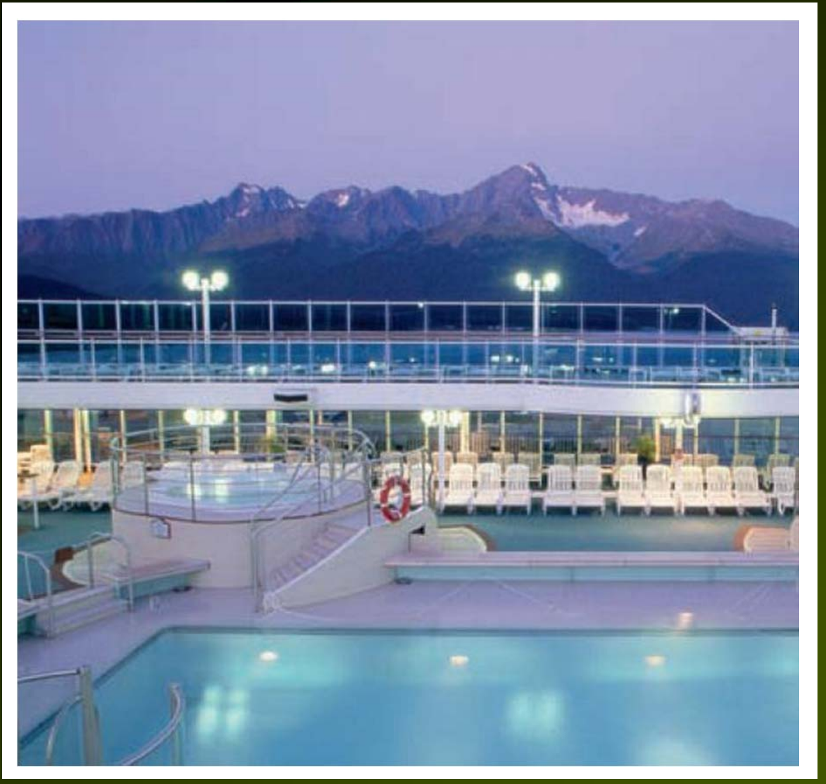
8 CALCULUS OF SEVERAL VARIABLES



8.1 Functions of Several Variables

Functions of Two Variables

A real-valued function of two variables f, consists of

1. A set A of ordered pairs of real numbers (x, y) called the domain of the function.

2. A rule that associates with each ordered pair in the domain of f one and only one real number, denoted by $z = f(x, y)$.

Example 1

Let *f* be the function defined by

$$f(x, y) = x + xy + y^2 + 2$$

Compute $f(0, 0)$, $f(1, 2)$, and $f(2, 1)$.

Solution:

$$f(0,0) = 0 + (0)(0) + 0^2 + 2 = 2$$

$$f(1,2) = 1 + (1)(2) + 2^2 + 2 = 9$$

$$f(2,1) = 2 + (2)(1) + 1^2 + 2 = 7$$

Functions of Two Variables

The domain of a function of two variables $f(x, y)$, is a set of ordered pairs of real numbers and may therefore be viewed as a subset of the xy-plane.

Example 2(a)

Find the domain of the function

$$f(x, y) = x^2 + y^2$$

Solution:

$f(x, y)$ is defined for all real values of x and y, so the domain of the function f is the set of all points (x, y) in the xy-plane.

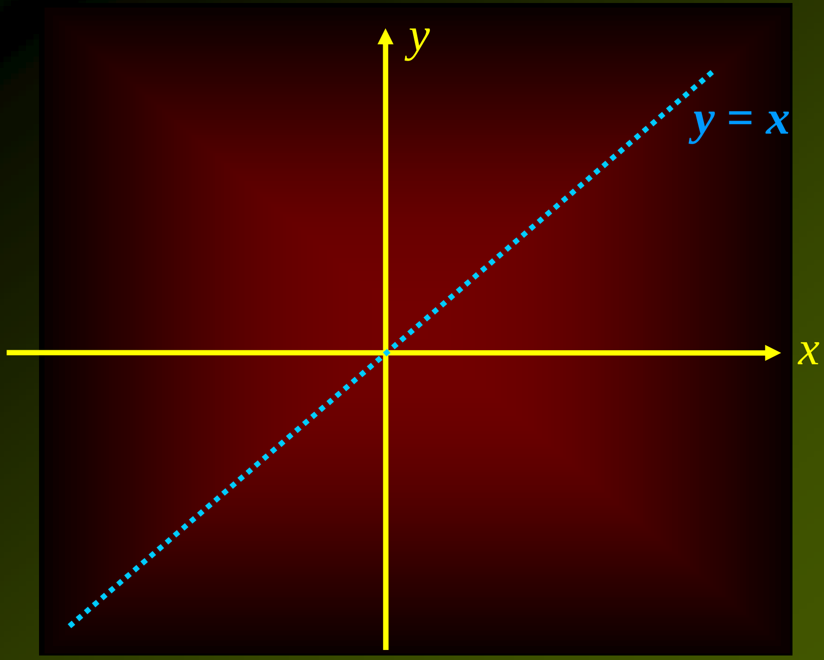
Example 2(b)

Find the domain of the function

$$g(x, y) = \frac{2}{x - y}$$

Solution:

$g(x, y)$ is defined for all $x \neq y$, so the domain of the function g is the set of all points (x, y) in the xy-plane except those lying on the $y = x$ line.



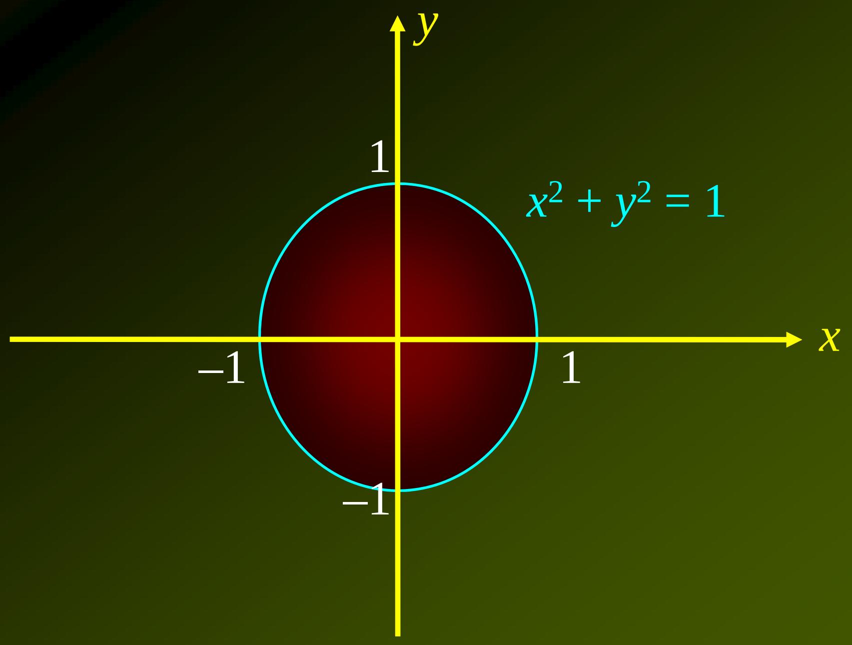
Example 2(c)

Find the domain of the function

$$h(x, y) = \sqrt{1 - x^2 - y^2}$$

Solution:

We require that $1 - x^2 - y^2 \geq 0$ or $x^2 + y^2 \leq 1$ which is the set of all points (x, y) lying on and inside the circle of radius 1 with center at the origin:



Applied Example 3 – *Revenue Functions*

Acrosonic manufactures a bookshelf loudspeaker system that may be bought fully assembled or in a kit. The demand equations that relate the unit price, p and q, to the quantities demanded weekly, x and y, of the assembled and kit versions of the loudspeaker systems are given by

$$p = 300 - \frac{1}{4}x - \frac{1}{8}y \quad \textbf{and} \quad q = 240 - \frac{1}{8}x - \frac{3}{8}y$$

a. What is the weekly total revenue function $R(x, y)$?

b. What is the domain of the function R?

Applied Example 3(a) – *Solution*

The weekly revenue from selling *x* units assembled speaker systems at *p* dollars per unit is given by *xp* dollars.

Similarly, the weekly revenue from selling *y* speaker kits at *q* dollars per unit is given by *yq* dollars.

Therefore, the weekly total revenue function *R* is given by

$$R(x, y) = xp + yq$$

$$= x\left(300 - \frac{1}{4}x - \frac{1}{8}y\right) + y\left(240 - \frac{1}{8}x - \frac{3}{8}y\right)$$

$$= -\frac{1}{4}x^2 - \frac{3}{8}y^2 - \frac{1}{4}xy + 300x + 240y$$

Applied Example 3(b) – *Solution* cont'd

To find the domain of the function *R*, note that the quantities *x*, *y*, *p*, and *q* must be nonnegative, which leads to the following system of linear inequalities:

$$300 - \frac{1}{4}x - \frac{1}{8}y \geq 0$$

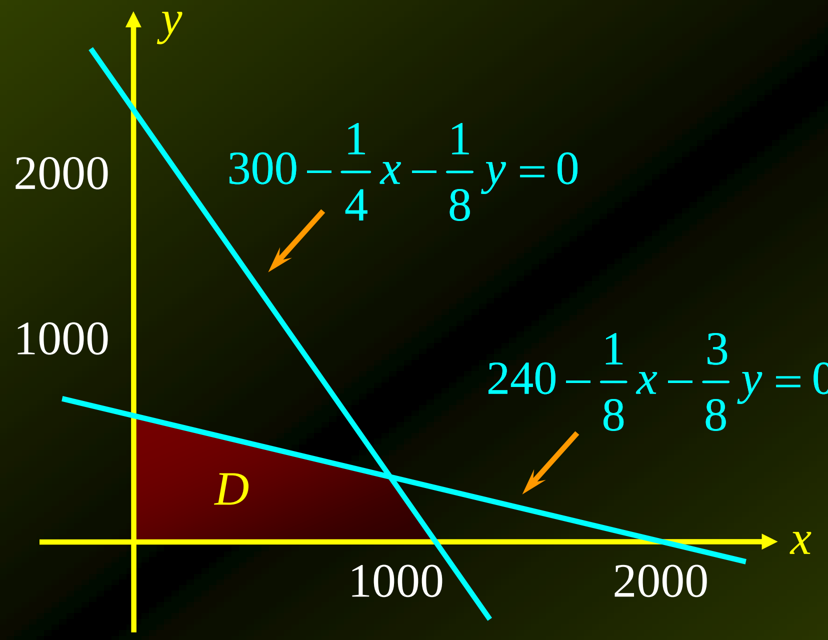
$$240 - \frac{1}{8}x - \frac{3}{8}y \geq 0$$

$$x \geq 0$$

$$y \geq 0$$

Applied Example 3(b) – *Solution* cont'd

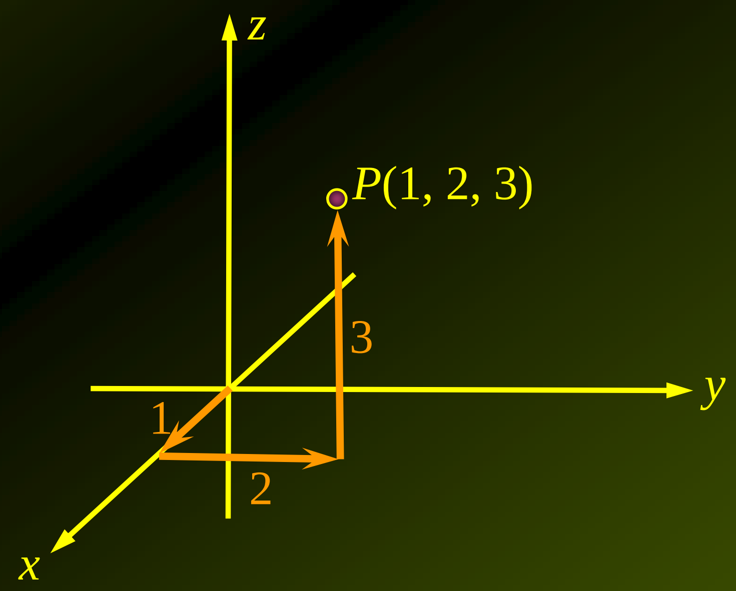
Thus, the graph of the domain is:



Graphs of Functions of Two Variables

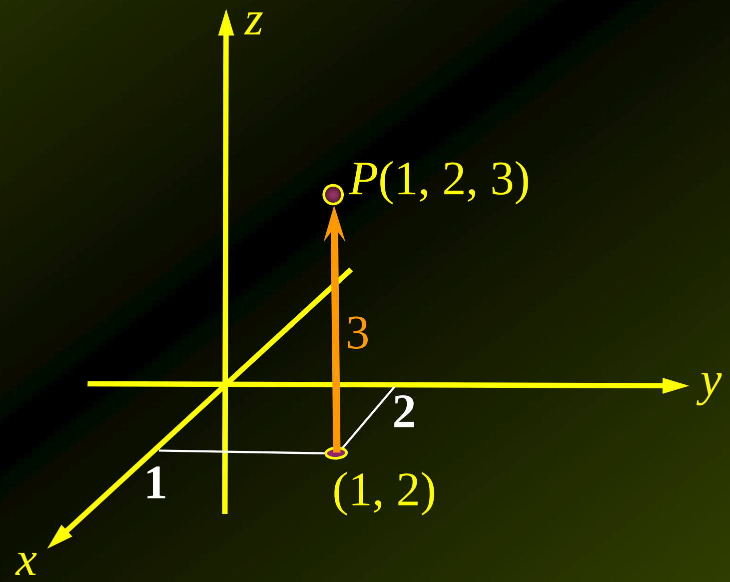
Consider the task of locating *P*(1, 2, 3) in 3-space:

One method to achieve this is to start at the origin and measure out from there, axis by axis:



Graphs of Functions of Two Variables

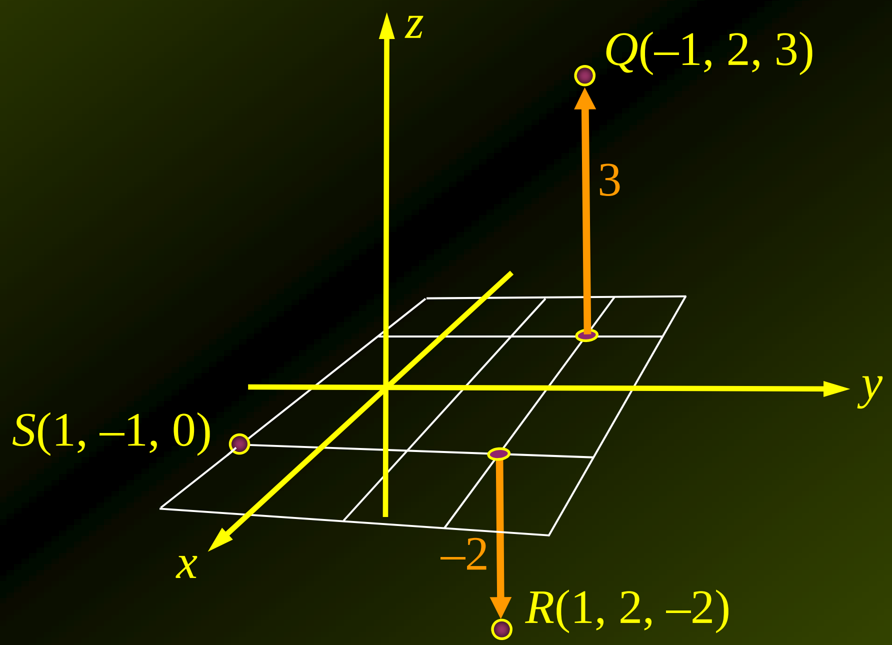
Another common method is to find the *xy* coordinate and from there elevate to the level of the *z* value:



Graphs of Functions of Two Variables

Locate the following points in 3-space:

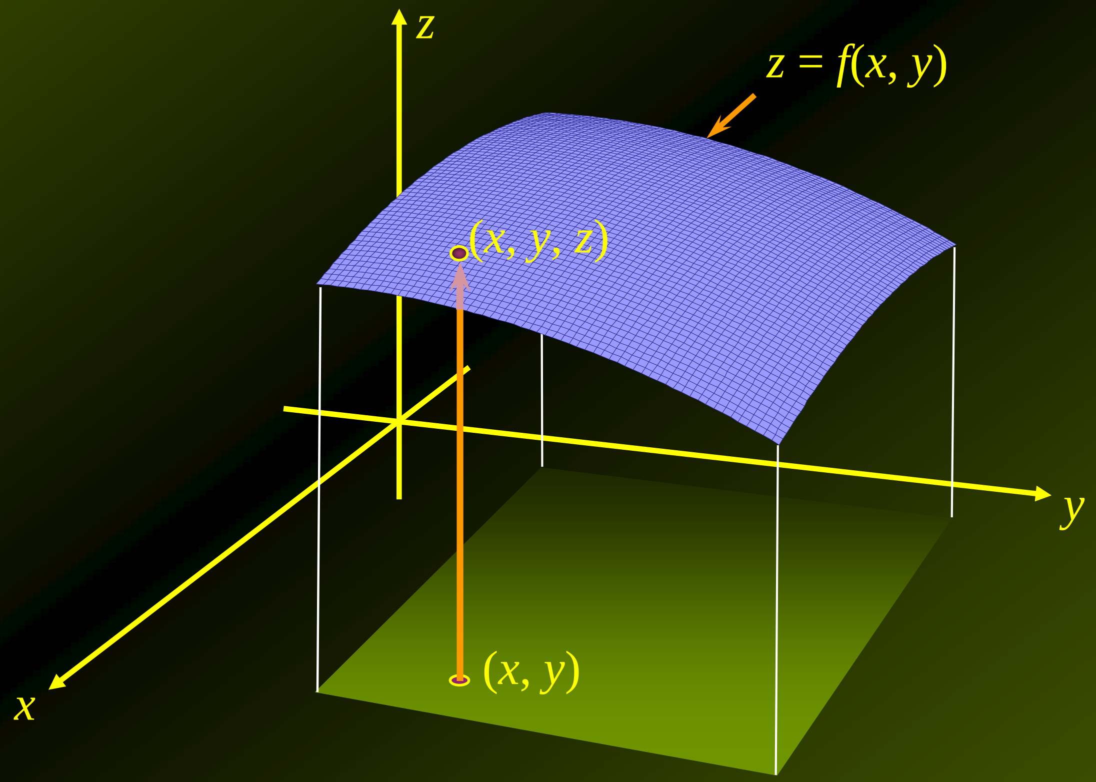
Q(–1, 2, 3), *R*(1, 2, –2), and *S*(1, –1, 0).



Graphs of Functions of Two Variables

The graph of a function in 3-space is a surface.

For every (x, y) in the domain of f, there is a z value on the surface.



Level Curves

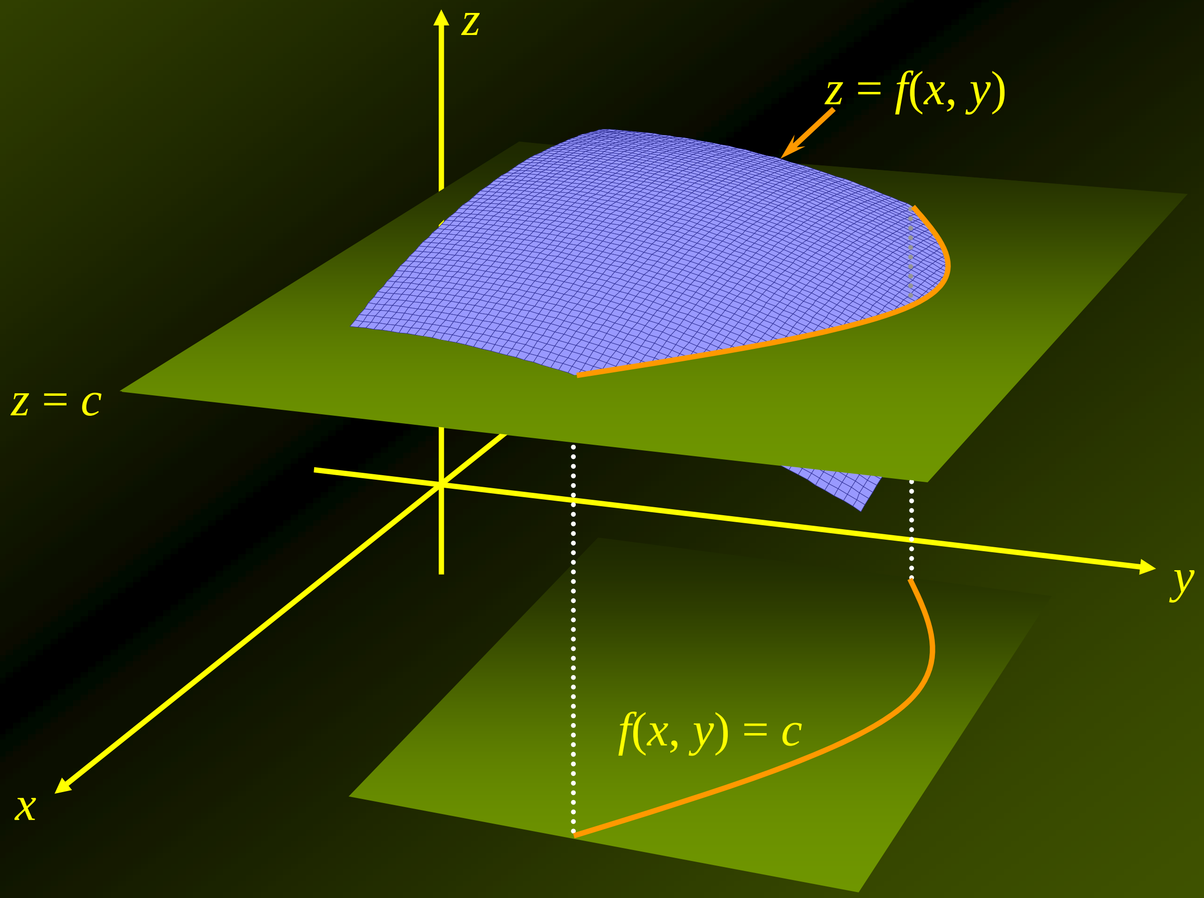
The graph of a function of two variables is often difficult to sketch.

It can therefore be useful to apply the method used to construct topographic maps.

This method is relatively easy to apply and conveys sufficient information to enable one to obtain a feel for the graph of the function.

Level Curves

In the 3-space graph we just saw, we can delineate the contour of the graph as it is cut by a $z = c$ plane:

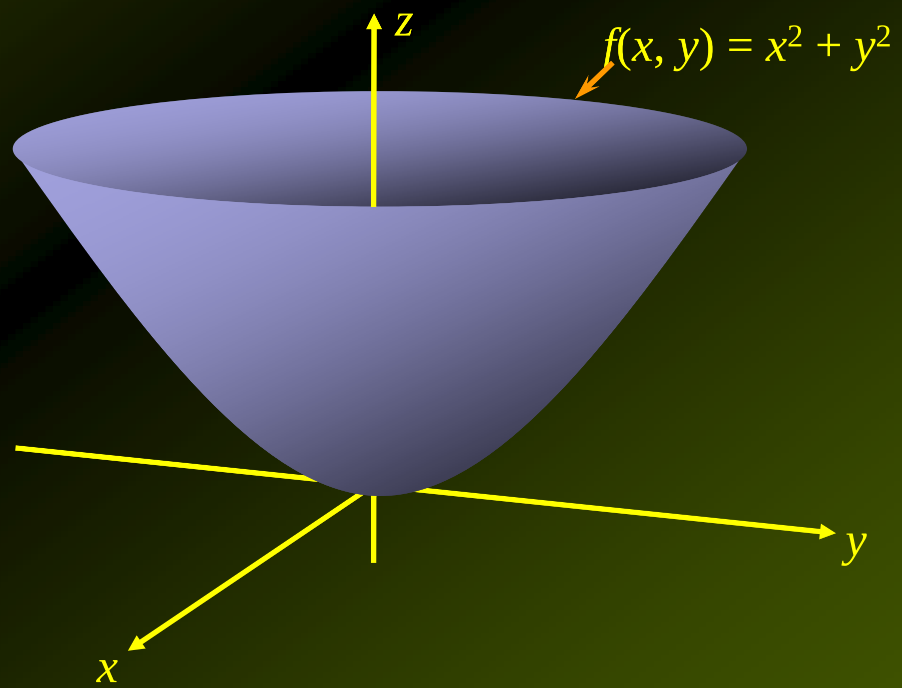


Example 5

Sketch a contour map of the function $f(x, y) = x^2 + y^2$.

Solution:

The function $f(x, y) = x^2 + y^2$ is a revolving parabola called a paraboloid.



Example 5 – *Solution*

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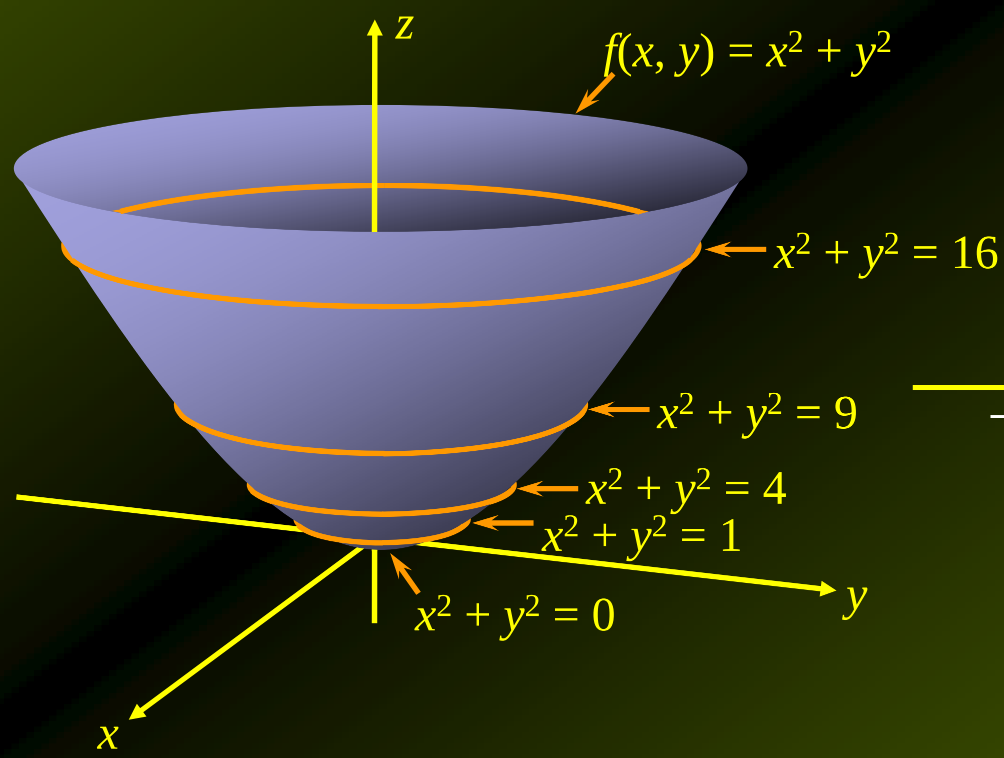
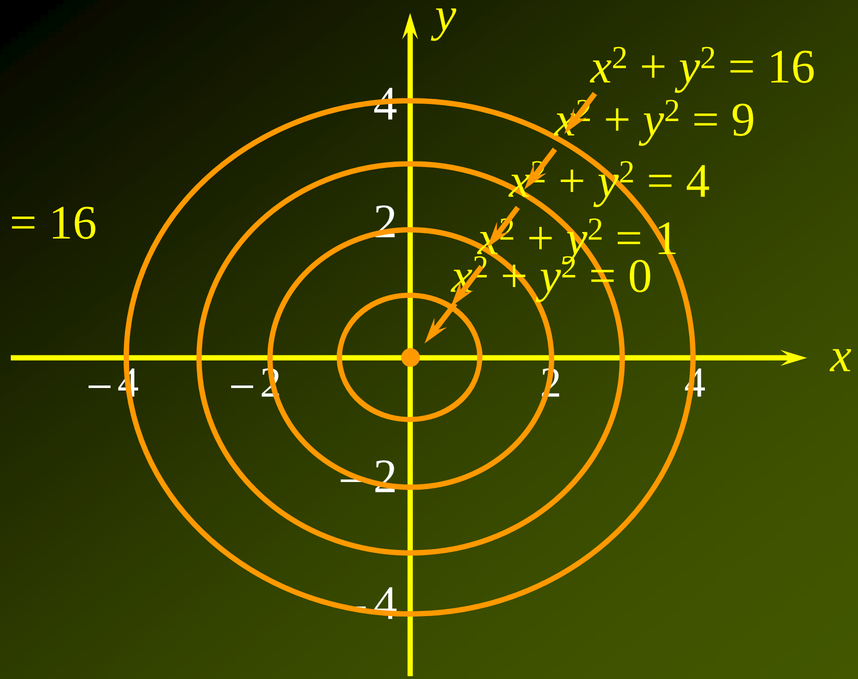
A level curve is the graph of the equation $x^2 + y^2 = c$, which describes a circle with radius $\sqrt{c}$.

Taking different values of c we obtain:





Example 6

Sketch level curves of the function $f(x, y) = 2x^2 - y$ corresponding to $z = -2, -1, 0, 1$, and 2.

Solution:

The level curves are the graphs of the equation $2x^2 - y = k$ or for $k = -2, -1, 0, 1$, and 2: