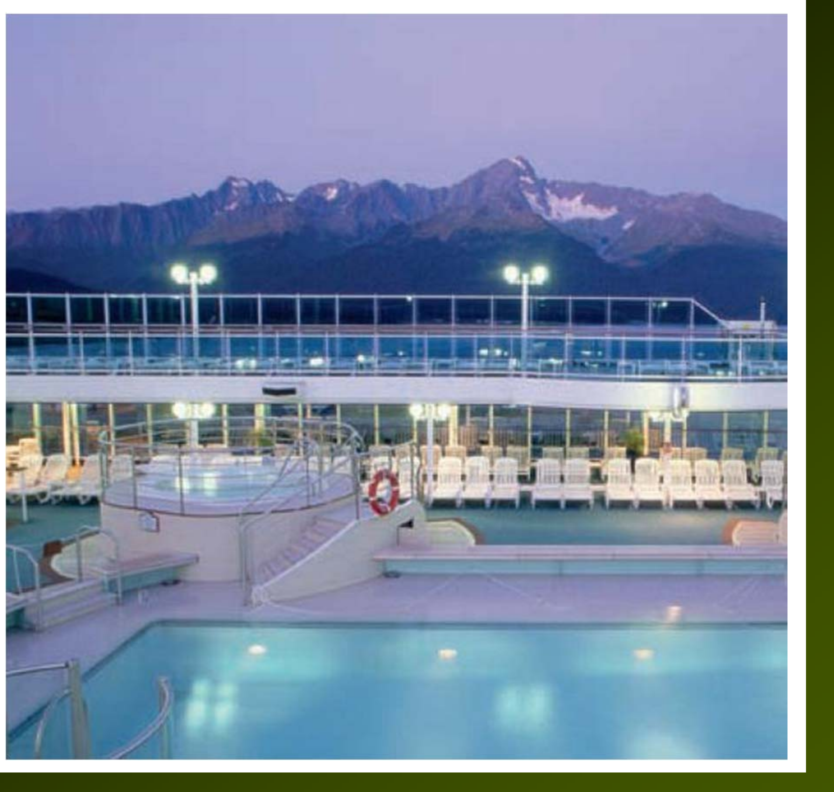
8

CALCULUS OF SEVERAL VARIABLES



8.2 Partial Derivatives

First Partial Derivatives

First Partial Derivatives of $f(x, y)$

- Suppose $f(x, y)$ is a function of two variables x and y.
- Then, the first partial derivative of f with respect to x at the point (x, y) is

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

provided the limit exists.

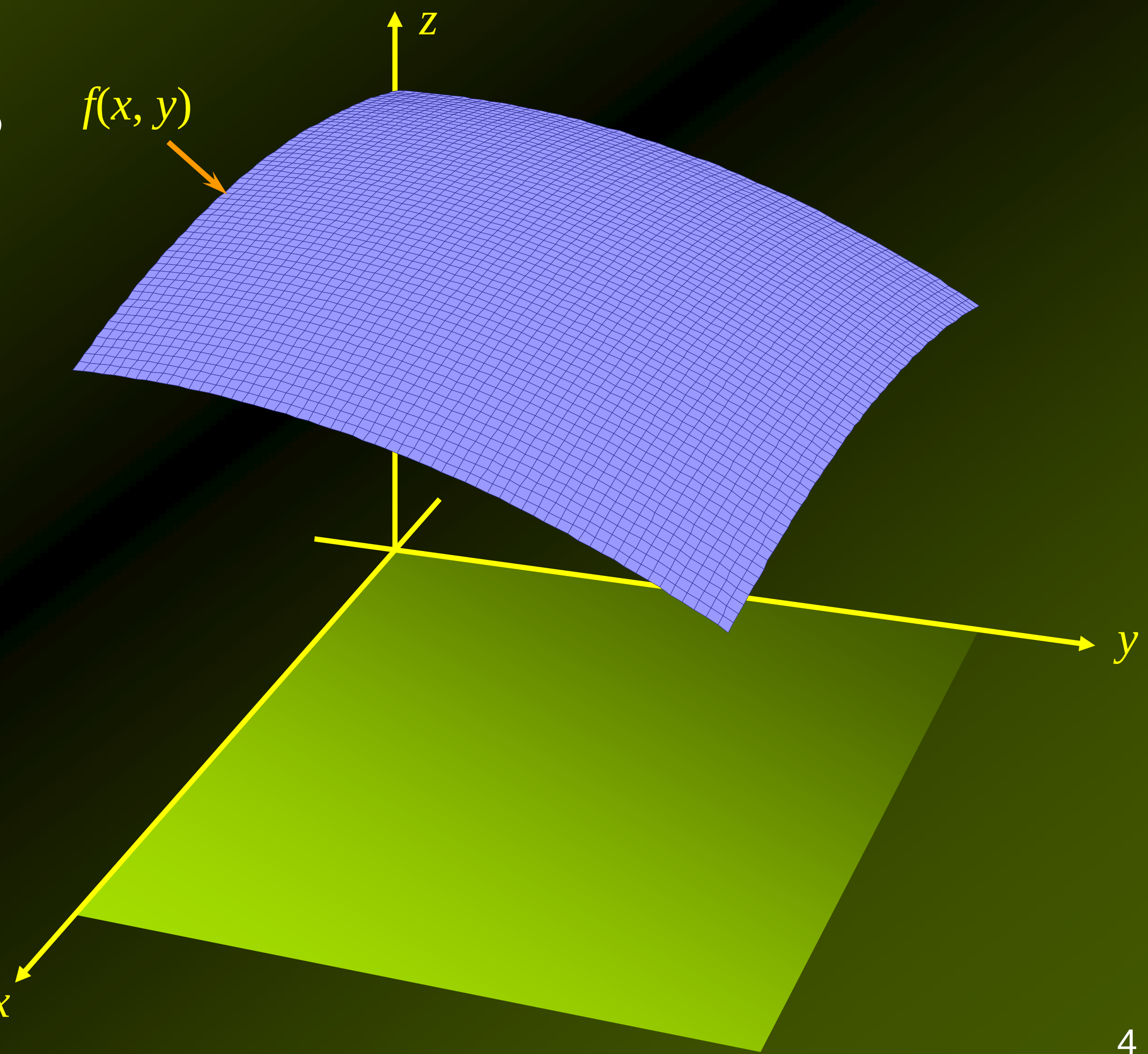
- The first partial derivative of f with respect to y at the point (x, y) is

$$\frac{\partial f}{\partial y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

provided the limit exists.

Geometric Interpretation of the Partial Derivative

What does $\dfrac{\partial f}{\partial x}$ **mean?**



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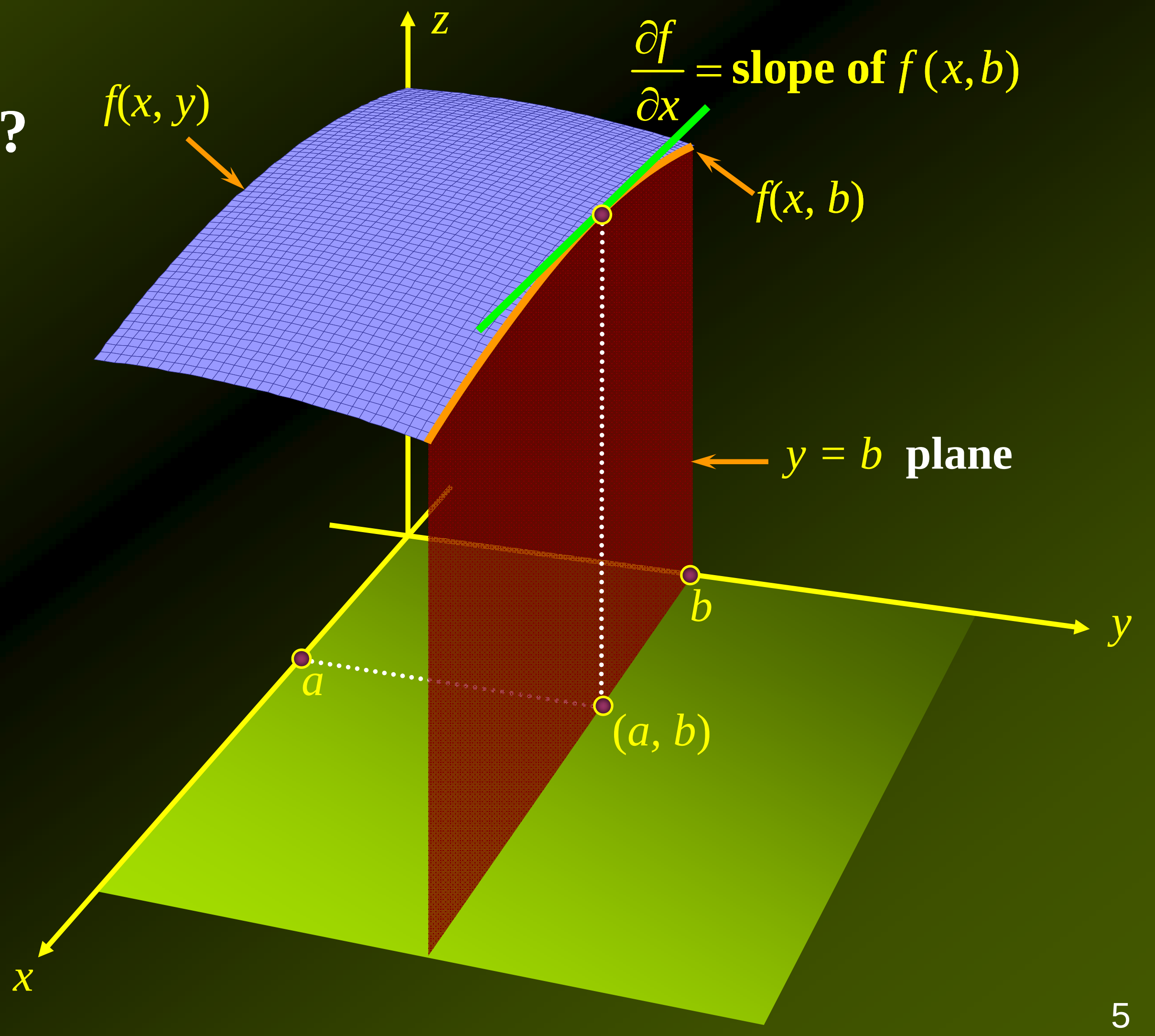
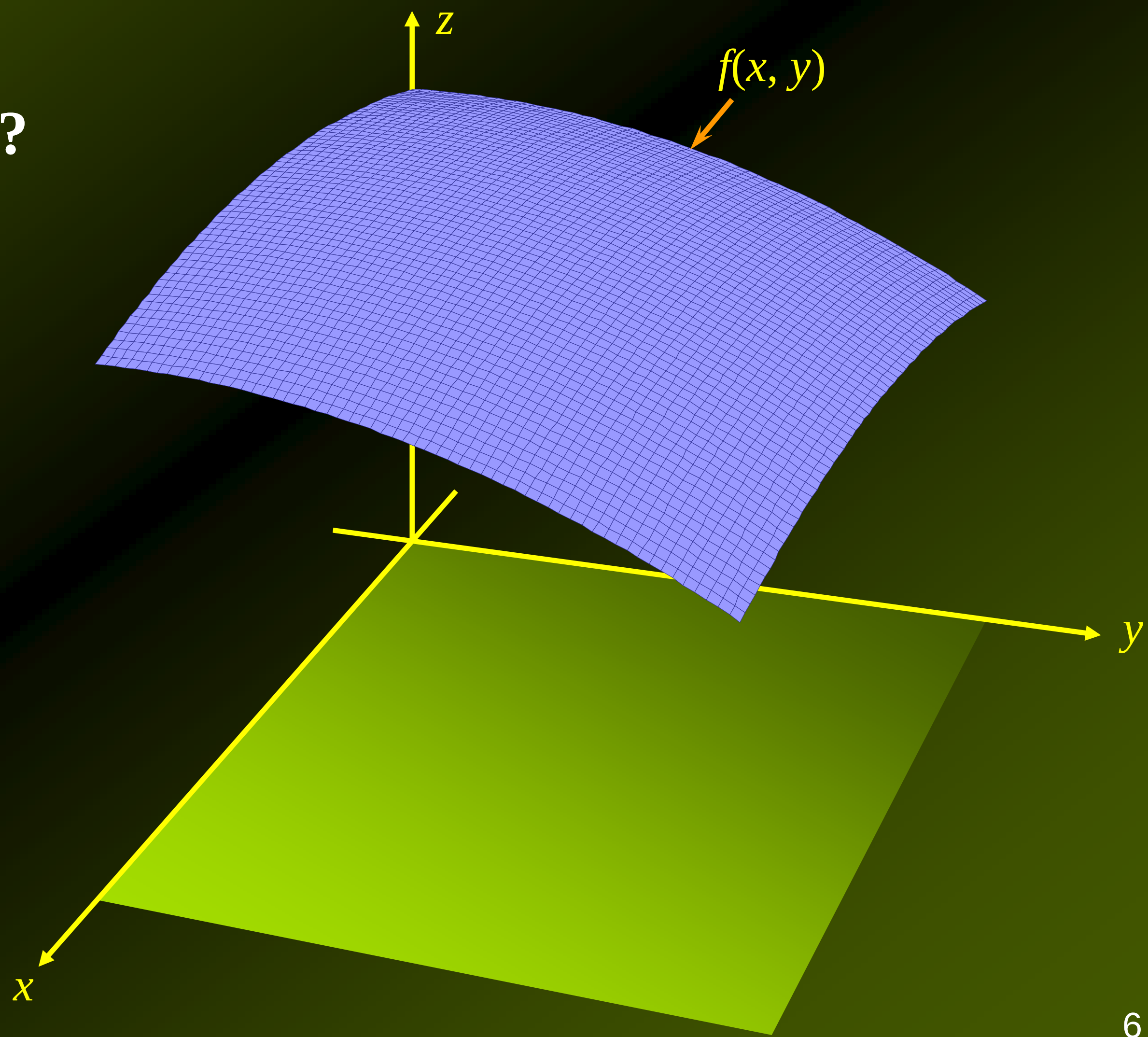


Geometric Interpretation of the Partial Derivative

What does $\frac{\partial f}{\partial y}$ **mean?**



Geometric Interpretation of the Partial Derivative

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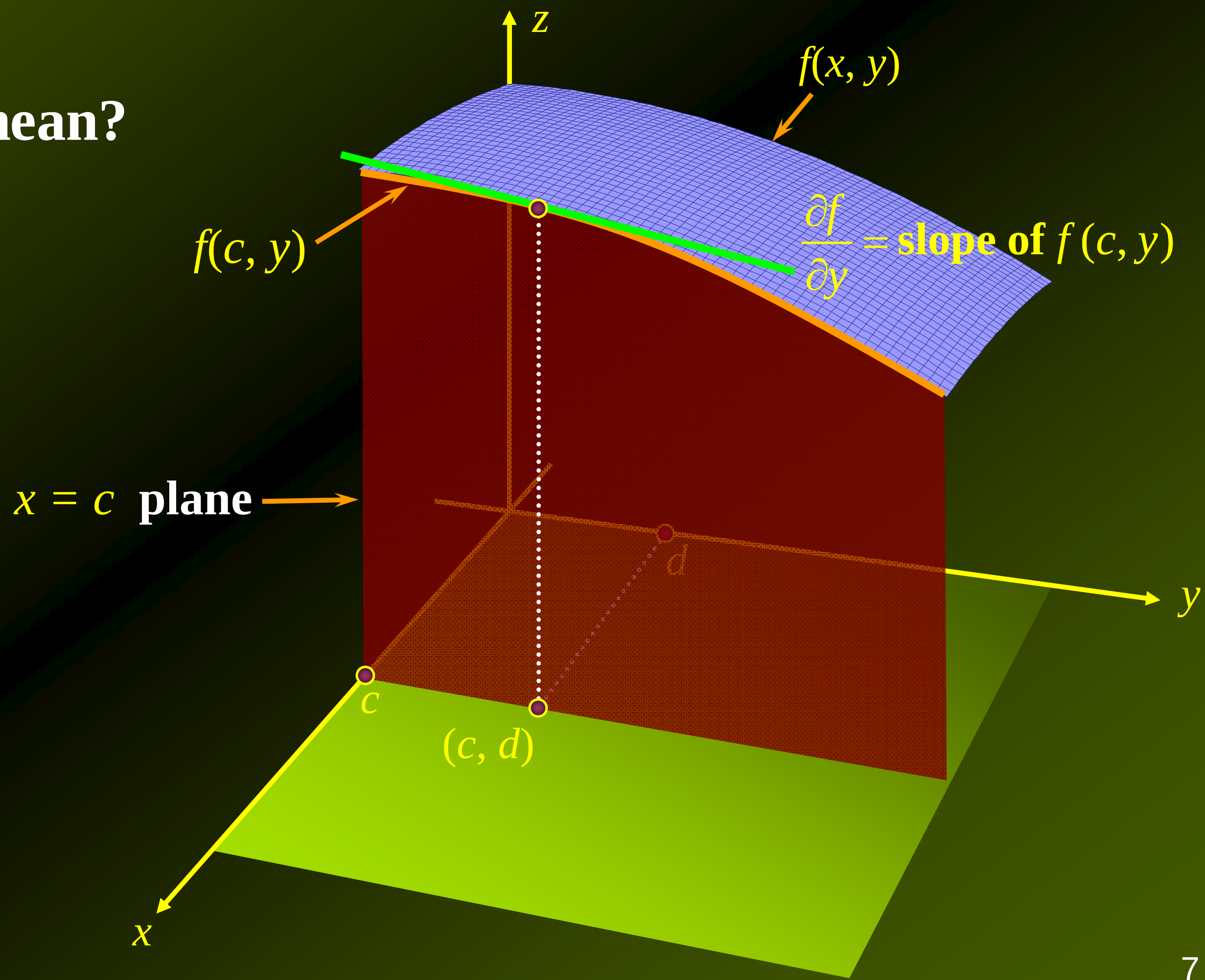

Example 1

Find the partial derivatives *∂f/∂x* and *∂f/∂y* of the function

$$f(x, y) = x^2 - xy^2 + y^3$$

Use the partials to determine the rate of change of *f* in the *x*-direction and in the *y*-direction at the point (1, 2).

Solution:

To compute *∂f/∂x*, think of the variable *y* as a constant and differentiate the resulting function of *x* with respect to *x*:

$$f(x, y) = x^2 - y^2x + y^3$$

$$\frac{\partial f}{\partial x} = 2x - y^2$$

Example 1 – *Solution*

cont'd

To compute *∂f/∂y*, think of the variable *x* as a constant and differentiate the resulting function of *y* with respect to *y*:

$$f(x, y) = x^2 - xy^2 + y^3$$

$$\frac{\partial f}{\partial y} = -2xy + 3y^2$$

The rate of change of *f* in the *x*-direction at the point (1, 2) is given by

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 2(1) - 2^2 = -2$$

Example 1 – *Solution*

cont'd

The rate of change of *f* in the *y*-direction at the point (1, 2) is given by

$$\left.\frac{\partial f}{\partial y}\right|_{(1,2)} = -2(1)(2) + 3(2)^2 = 8$$

Example 2(a)

Find the first partial derivatives of the function

$$w(x, y) = \frac{xy}{x^2 + y^2}$$

Solution:

To compute *∂w/∂x*, think of the variable *y* as a constant and differentiate the resulting function of *x* with respect to *x*:

$$w(x, y) = \frac{xy}{x^2 + y^2}$$

$$\frac{\partial w}{\partial x} = \frac{(x^2 + y^2)y - xy(2x)}{(x^2 + y^2)^2}$$

$$= \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$

Example 2(a) – *Solution*

cont'd

To compute $\partial w/\partial y$, think of the variable x as a constant and differentiate the resulting function of y with respect to y:

$$w(x, y) = \frac{xy}{x^2 + y^2}$$

$$\frac{\partial w}{\partial y} = \frac{(x^2 + y^2)x - xy(2y)}{(x^2 + y^2)^2}$$

$$= \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}$$

Example 2(b)

Find the first partial derivatives of the function

$$g(s,t) = (s^2 - st + t^2)^5$$

Solution:

To compute $\partial g/\partial s$, think of the variable t as a constant and differentiate the resulting function of s with respect to s:

$$g(s,t) = (s^2 - st + t^2)^5$$

$$\frac{\partial g}{\partial s} = 5(s^2 - st + t^2)^4 \cdot (2s - t)$$

$$= 5(2s - t)(s^2 - st + t^2)^4$$

Example 2(b) – *Solution*

cont'd

To compute $\partial g/\partial t$, think of the variable s as a constant and differentiate the resulting function of t with respect to t:

$$g(s,t) = (s^2 - st + t^2)^5$$

$$\frac{\partial g}{\partial t} = 5(s^2 - st + t^2)^4 \cdot (-s + 2t)$$

$$= 5(2t - s)(s^2 - st + t^2)^4$$

Example 2(c)

Find the first partial derivatives of the function

$$h(u,v) = e^{u^2 - v^2}$$

Solution:

To compute $\partial h/\partial u$, think of the variable v as a constant and differentiate the resulting function of u with respect to u:

$$h(u,v) = e^{u^2 - v^2}$$

$$\frac{\partial h}{\partial u} = e^{u^2 - v^2} \cdot 2u$$

$$= 2ue^{u^2 - v^2}$$

Example 2(c) – *Solution*

cont'd

To compute $\partial h/\partial v$, think of the variable u as a constant and differentiate the resulting function of v with respect to v:

$$h(u, v) = e^{u^2 - v^2}$$

$$\frac{\partial h}{\partial u} = e^{u^2 - v^2} \cdot (-2v)$$

$$= -2ve^{u^2 - v^2}$$

Example 3

Find the first partial derivatives of the function

$$w = f(x, y, z) = xyz - xe^{yz} + x \ln y$$

Solution:

Here we have a function of three variables, x, y, and z, and we are required to compute

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$$

For short, we can label these first partial derivatives respectively f_x, f_y, and f_z.

Example 3 – *Solution*

cont'd

To find f_x, think of the variables y and z as a constant and differentiate the resulting function of x with respect to x:

$$w = f(x, y, z) = xyz - xe^{yz} + x \ln y$$

$$f_x = yz - e^{yz} + \ln y$$

To find f_y, think of the variables x and z as a constant and differentiate the resulting function of y with respect to y:

$$w = f(x, y, z) = xyz - xe^{yz} + x \ln y$$

$$f_y = xz - xze^{yz} + \frac{x}{y}$$

Example 3 – *Solution*

cont'd

To find f_z, think of the variables x and y as a constant and differentiate the resulting function of z with respect to z:

$$w = f(x, y, z) = xyz - xe^{yz} + x \ln y$$

$$f_z = xy - xye^{yz}$$

The Cobb-Douglas Production Function

The Cobb-Douglass Production Function is of the form

$$f(x, y) = ax^b y^{1-b} \qquad (0 < b < 1)$$

where

a and b are positive constants,

x stands for the cost of labor,

y stands for the cost of capital equipment, and

f measures the output of the finished product.

The Cobb-Douglas Production Function

The Cobb-Douglass Production Function is of the form

$$f(x, y) = ax^b y^{1-b} \qquad (0 < b < 1)$$

The first partial derivative f_x is called the marginal productivity of labor.

- It measures the rate of change of production with respect to the amount of money spent on labor, with the level of capital kept constant.

The first partial derivative f_y is called the marginal productivity of capital.

- It measures the rate of change of production with respect to the amount of money spent on capital, with the level of labor kept constant.

Applied Example 4 – *Marginal Productivity*

A certain country's production in the early years following World War II is described by the function

$$f(x, y) = 30x^{2/3}y^{1/3}$$

when x units of labor and y units of capital were used.

Compute f_x and f_y.

Find the marginal productivity of labor and the marginal productivity of capital when the amount expended on labor and capital was 125 units and 27 units, respectively.

Should the government have encouraged capital investment rather than increase expenditure on labor to increase the country's productivity?

Applied Example 4 – *Solution*

The first partial derivatives are

$$f_x = 30 \cdot \frac{2}{3} x^{-1/3} y^{1/3} = 20\left(\frac{y}{x}\right)^{1/3}$$

$$f_y = 30x^{2/3} \cdot \frac{1}{3} y^{-2/3} = 10\left(\frac{x}{y}\right)^{2/3}$$

The required marginal productivity of labor is given by

$$f_x(125, 27) = 20\left(\frac{27}{125}\right)^{1/3} = 20\left(\frac{3}{5}\right) = 12$$

or 12 units of output per unit increase in labor expenditure (keeping capital constant).

Applied Example 4 – *Solution*

cont'd

The required marginal productivity of capital is given by

$$f_y(125, 27) = 10\left(\frac{125}{27}\right)^{2/3} = 10\left(\frac{25}{9}\right) = 27\tfrac{7}{9}$$

or $27\,{}^{7}\!/_{9}$ units of output per unit increase in capital expenditure (keeping labor constant).

The government should definitely have encouraged capital investment.

A unit increase in capital expenditure resulted in a much faster increase in productivity than a unit increase in labor: $27\,{}^{7}\!/_{9}$ versus 12 per unit of investment, respectively.

Second Order Partial Derivatives

The first partial derivatives $f_x(x, y)$ and $f_y(x, y)$ of a function $f(x, y)$ of two variables x and y are also functions of x and y.

As such, we may differentiate each of the functions f_x and f_y to obtain the second-order partial derivatives of f.

Second Order Partial Derivatives

Differentiating the function f_x with respect to x leads to the second partial derivative

$$f_{xx} \equiv \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(f_x)$$

But the function f_x can also be differentiated with respect to y leading to a different second partial derivative

$$f_{xy} \equiv \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(f_x)$$

Second Order Partial Derivatives

Similarly, differentiating the function f_y with respect to y leads to the second partial derivative

$$f_{yy} \equiv \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(f_y)$$

Finally, the function f_y can also be differentiated with respect to x leading to the second partial derivative

$$f_{yx} \equiv \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(f_y)$$

Second Order Partial Derivatives

Thus, four second-order partial derivatives can be obtained of a function of two variables:

$$f \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial f}{\partial x}$$

$$\frac{\partial f}{\partial x} \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial f}{\partial x} \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial f}{\partial y}$$

$$\frac{\partial f}{\partial y} \xrightarrow{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial f}{\partial y} \xrightarrow{\frac{\partial}{\partial y}} \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

When both are continuous

Example 6

Find the second-order partial derivatives of the function

$$f(x, y) = x^3 - 3x^2y + 3xy^2 + y^2$$

Solution:

First, calculate f_x and use it to find f_{xx} and f_{xy}:

$$f_x = \frac{\partial}{\partial x}(x^3 - 3x^2y + 3xy^2 + y^2)$$

$$= 3x^2 - 6xy + 3y^2$$

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 - 6xy + 3y^2)$$

$$= 6x - 6y$$

$$= 6(x - y)$$

Example 6 – *Solution*

cont'd

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 - 6xy + 3y^2)$$

$$= -6x + 6y$$

$$= 6(y - x)$$

Then, calculate f_y and use it to find f_{yx} and f_{yy}:

$$f_y = \frac{\partial}{\partial y}(x^3 - 3x^2 y + 3xy^2 + y^2)$$

$$= -3x^2 + 6xy + 2y$$

Example 6 – *Solution*

cont'd

$$f_{yx} = \frac{\partial}{\partial x}(-3x^2 + 6xy + 2y)$$

$$= -6x + 6y$$

$$= 6(y - x)$$

$$f_{yy} = \frac{\partial}{\partial y}(-3x^2 + 6xy + 2y)$$

$$= 6x + 2$$

$$= 2(3x + 1)$$

Example 7

Find the second-order partial derivatives of the function

$$f(x, y) = e^{xy^2}$$

Solution:

First, calculate f_x and use it to find f_{xx} and f_{xy}:

$$f_x = \frac{\partial}{\partial x}(e^{xy^2})$$

$$= y^2 e^{xy^2}$$

$$f_{xx} = \frac{\partial}{\partial x}(y^2 e^{xy^2})$$

$$= y^4 e^{xy^2}$$

Example 7 – *Solution*

cont'd

$$f_{xy} = \frac{\partial}{\partial y}(y^2 e^{xy^2})$$

$$= 2ye^{xy^2} + 2xy^3 e^{xy^2}$$

$$= 2ye^{xy^2}(1 + xy^2)$$

Then, calculate f_y and use it to find f_{yx} and f_{yy}:

$$f_y = \frac{\partial}{\partial y}(e^{xy^2})$$

$$= 2xye^{xy^2}$$

Example 7 – *Solution*

cont'd

$$f_{yx} = \frac{\partial}{\partial x}(2xye^{xy^2})$$

$$= 2ye^{xy^2} + 2xy^3e^{xy^2}$$

$$= 2ye^{xy^2}(1 + xy^2)$$

$$f_{yy} = \frac{\partial}{\partial y}(2xye^{xy^2})$$

$$= 2xe^{xy^2} + (2xy)(2xy)e^{xy^2}$$

$$= 2xe^{xy^2}(1 + 2xy^2)$$