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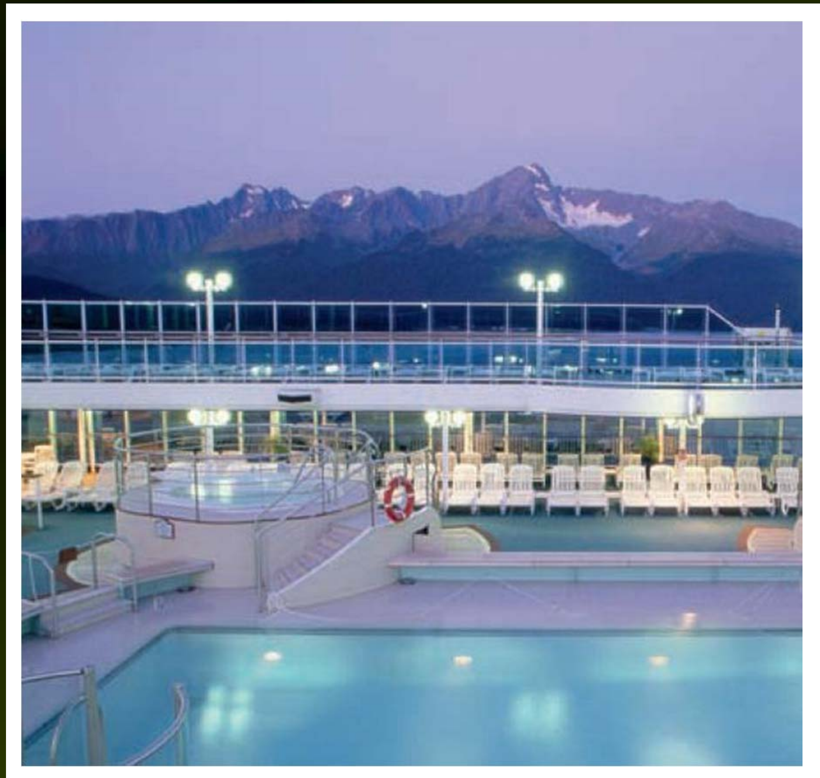
CALCULUS OF SEVERAL VARIABLES



8.5 Constrained Maxima and Minima and the Method of Lagrange Multipliers

Constrained Maxima and Minima

In many practical optimization problems, we must maximize or minimize a function in which the independent variables are subjected to certain further constraints.

We shall discuss a powerful method for determining relative extrema of a function $f(x, y)$ whose independent variables x and y are required to satisfy one or more constraints of the form $g(x, y) = 0$.

Example 1

Find the relative minimum of $f(x, y) = 2x^2 + y^2$ subject to the constraint $g(x, y) = x + y - 1 = 0$.

Solution:

Solving the constraint equation for y explicitly in terms of x, we obtain

$$y = -x + 1$$

Substituting this value of y into $f(x, y)$ results in a function of x,

$$h(x) = 2x^2 + (-x+1)^2 = 3x^2 - 2x + 1$$

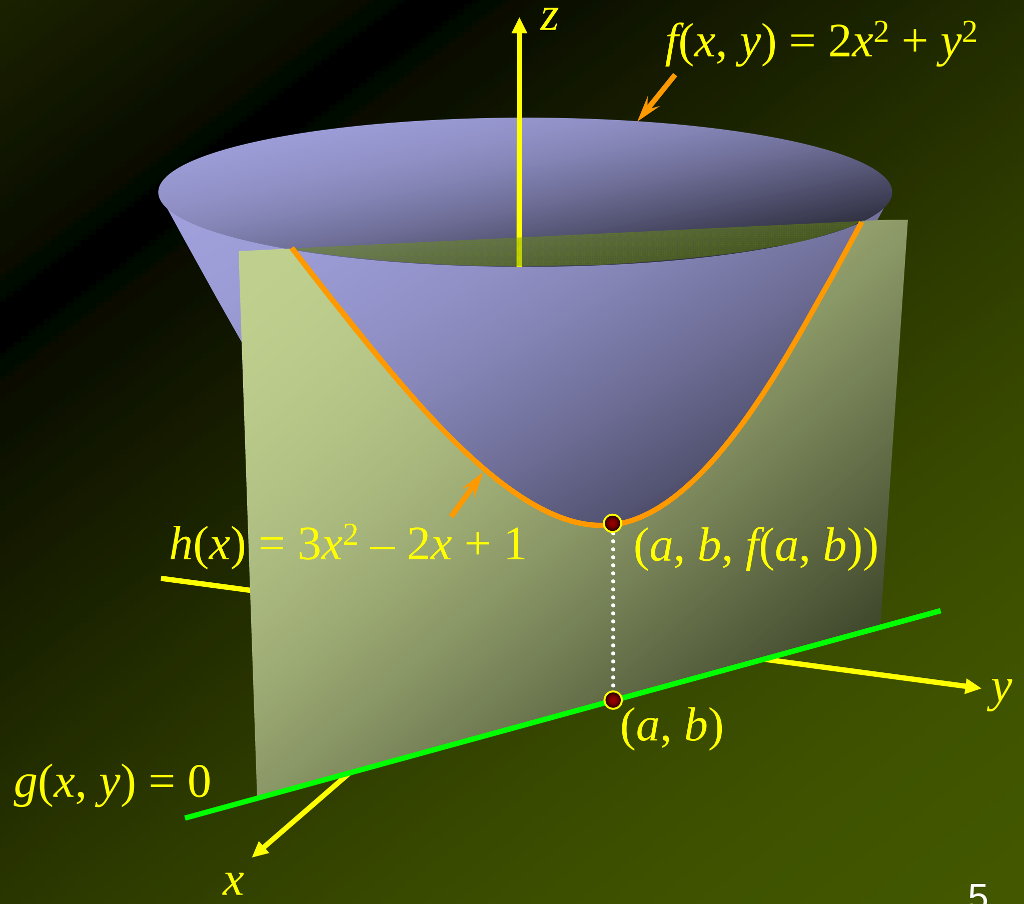
Example 1 – *Solution*

cont'd

We have

$$h(x) = 3x^2 - 2x + 1.$$

The function h describes the curve lying on the graph of f on which the constrained relative minimum point (a, b) of f occurs:



Example 1 – *Solution*

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To find this point (a, b), we determine the relative extrema of a function of one variable:

$$h'(x) = 6x - 2 = 2(3x - 1)$$

Setting $h' = 0$ gives $x = \frac{1}{3}$ as the sole critical point of h.

Next, we find $h''(x) = 6$ and, in particular, $h''(\frac{1}{3}) = 6 > 0$.

Therefore, by the second derivative test, the point gives rise to a relative minimum of h.

Example 1 – *Solution*

cont'd

Substitute $x = \frac{1}{3}$ into the constraint equation $x + y - 1 = 0$ to get $y = \frac{1}{3}$.

Thus, the point $\left(\frac{1}{3}, \frac{2}{3}\right)$ gives rise to the required constrained relative minimum of f.

Since $f\left(\frac{1}{3}, \frac{2}{3}\right) = 2\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{2}{3}$, the required constrained relative minimum value of f is $\frac{2}{3}$ at the point $\left(\frac{1}{3}, \frac{2}{3}\right)$.

It may be shown that $\frac{2}{3}$ is in fact a constrained absolute minimum value of f.

The Method of Lagrange Multipliers

To find the relative extrema of the function $f(x, y)$ subject to the constraint $g(x, y) = 0$ (assuming that these extreme values exist),

1. Form an auxiliary function

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

   called the Lagrangian function (the variable λ is called the Lagrange multiplier).

2. Solve the system that consists of the equations

$$F_x = 0 \qquad F_y = 0 \qquad F_\lambda = 0$$

   for all values of x, y, and λ.

3. The solutions found in step 2 are candidates for the extrema of f.

Example 2

Using the method of Lagrange multipliers, find the relative minimum of the function $f(x, y) = 2x^2 + y^2$ subject to the constraint $x + y = 1$.

Solution:

Write the constraint equation

$$x + y = 1$$

in the form

$$g(x, y) = x + y - 1 = 0$$

Then, form the Lagrangian function

$$\begin{aligned} F(x, y, \lambda) &= f(x, y) + \lambda g(x, y) \\ &= (2x^2 + y^2) + \lambda(x + y - 1) \end{aligned}$$

Example 2 – *Solution*

cont'd

We have

$$F = 2x^2 + y^2 + \lambda(x + y - 1)$$

To find the critical point(s) of the function *F*, solve the system composed of the equations

$$F_x = 4x + \lambda = 0 \qquad F_y = 2y + \lambda = 0 \qquad F_\lambda = x + y - 1 = 0$$

Solving the first and second equations for *x* and *y* in terms of *λ*, we obtain

$$x = -\frac{1}{4}\lambda \quad \text{and} \quad y = -\frac{1}{2}\lambda$$

Which, upon substitution into the third equation yields

$$-\frac{1}{4}\lambda - \frac{1}{2}\lambda - 1 = 0 \quad \text{or} \quad \lambda = -\frac{4}{3}$$

Example 2 – *Solution*

cont'd

Substituting $\lambda = -\frac{4}{3}$ into $x = -\frac{1}{4}\lambda$ and $y = -\frac{1}{2}\lambda$

yields $x = -\frac{1}{4}(-\frac{4}{3}) = \frac{1}{3}$ and $y = -\frac{1}{2}(-\frac{4}{3}) = \frac{2}{3}$

Therefore, $x = \frac{1}{3}$ and $y = \frac{2}{3}$, and $\left(\frac{1}{3}, \frac{2}{3}\right)$ results in a constrained minimum of the function *f*.

Applied Example 5 – *Designing a Cruise-Ship Pool*

The operators of the Viking Princess, a luxury cruise liner, are contemplating the addition of another swimming pool to the ship. The chief engineer has suggested that an area of the form of an ellipse located in the rear of the promenade deck would be suitable for this purpose. It has been determined that the shape of the ellipse may be described by the equation

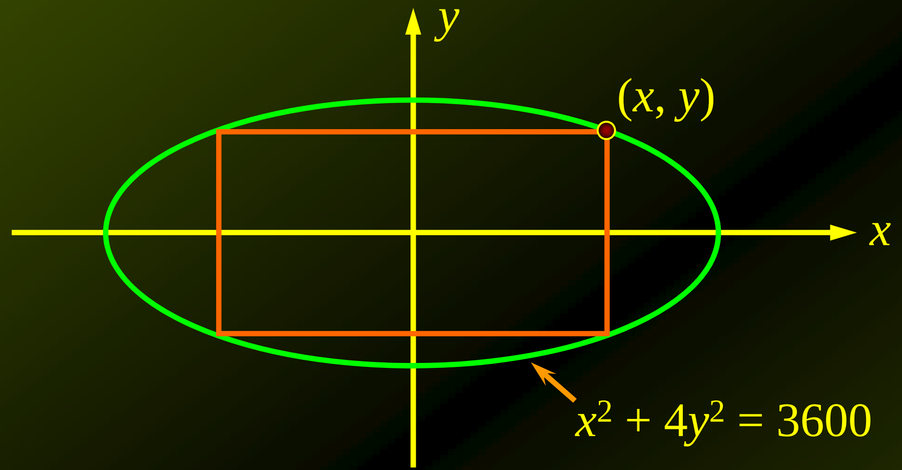
$$x^2 + 4y^2 = 3600$$

where x and y are measured in feet.

Viking's operators would like to know the dimensions of the rectangular pool with the largest possible area that would meet these requirements.

Applied Example 5 – *Solution*

We want to maximize the area of the rectangle that will fit the ellipse:



Letting the sides of the rectangle be $2x$ and $2y$ feet, we see that the area of the rectangle is $A = 4xy$.

Furthermore, the point (x, y) must be constrained to lie on the ellipse so that it satisfies the equation $x^2 + 4y^2 = 3600$.

Applied Example 5 – *Solution*

cont'd

Thus, the problem is equivalent to the problem of maximizing the function

$$f(x, y) = 4xy$$

subject to the constraint

$$g(x, y) = x^2 + 4y^2 - 3600 = 0$$

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda g(x, y) = 4xy + \lambda(x^2 + 4y^2 - 3600)$$

To find the critical points of F, we solve the system of equations

$$F_x = 4y + 2\lambda x = 0$$

$$F_y = 4x + 8\lambda y = 0$$

$$F_\lambda = x^2 + 4y^2 - 3600 = 0$$

Applied Example 5 – *Solution*

cont'd

Solving the first equation for λ, we obtain $\lambda = -\frac{2y}{x}$

Which, substituting into the second equation, yields

$$4x + 8\left(-\frac{2y}{x}\right)y = 0 \quad \text{or} \quad x^2 - 4y^2 = 0$$

Solving for x yields $x = \pm 2y$.

Substituting $x = \pm 2y$ into the third equation, we have

$$4y^2 + 4y^2 - 3600 = 0$$

Applied Example 5 – *Solution*

cont'd

Which, upon solving for *y* yields

$$y = \pm\sqrt{450} = \pm 15\sqrt{2}$$

The corresponding values of *x* are

$$x = \pm 2y = \pm 2(15\sqrt{2}) = \pm 30\sqrt{2}$$

Since both *x* and *y* must be nonnegative, we have

$$x = 30\sqrt{2} \quad \text{and} \quad y = 15\sqrt{2}$$

or approximately 42 feet × 85 feet.