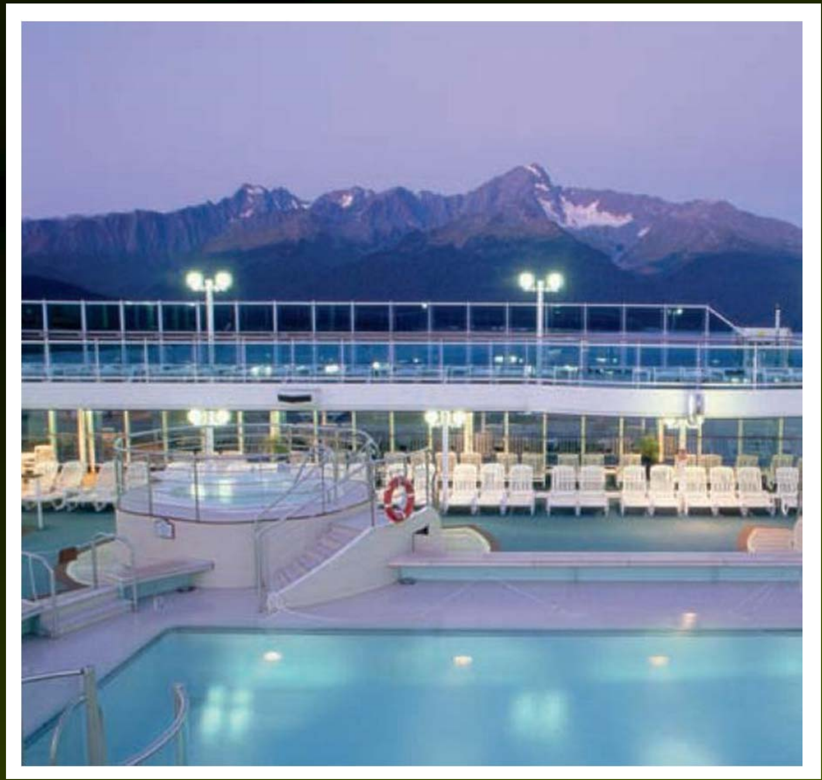
8 CALCULUS OF SEVERAL VARIABLES



8.6 Total Differentials

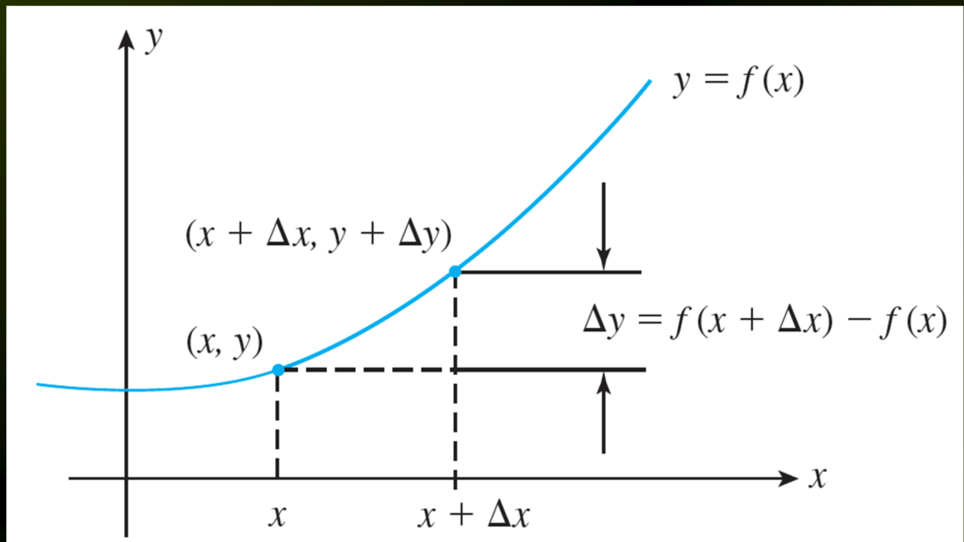
Increments

Increments

Recall that if f is a function of one variable defined by $y = f(x)$, then the *increment* in y is defined to be

$$\Delta y = f(x + \Delta x) - f(x)$$

where Δx is an increment in x (Figure 30a).



(a) The increment Δy is the change in y as x changes from x to $x + \Delta x$.

Figure 30

Increments

The increment of a function of two or more variables is defined in an analogous manner.

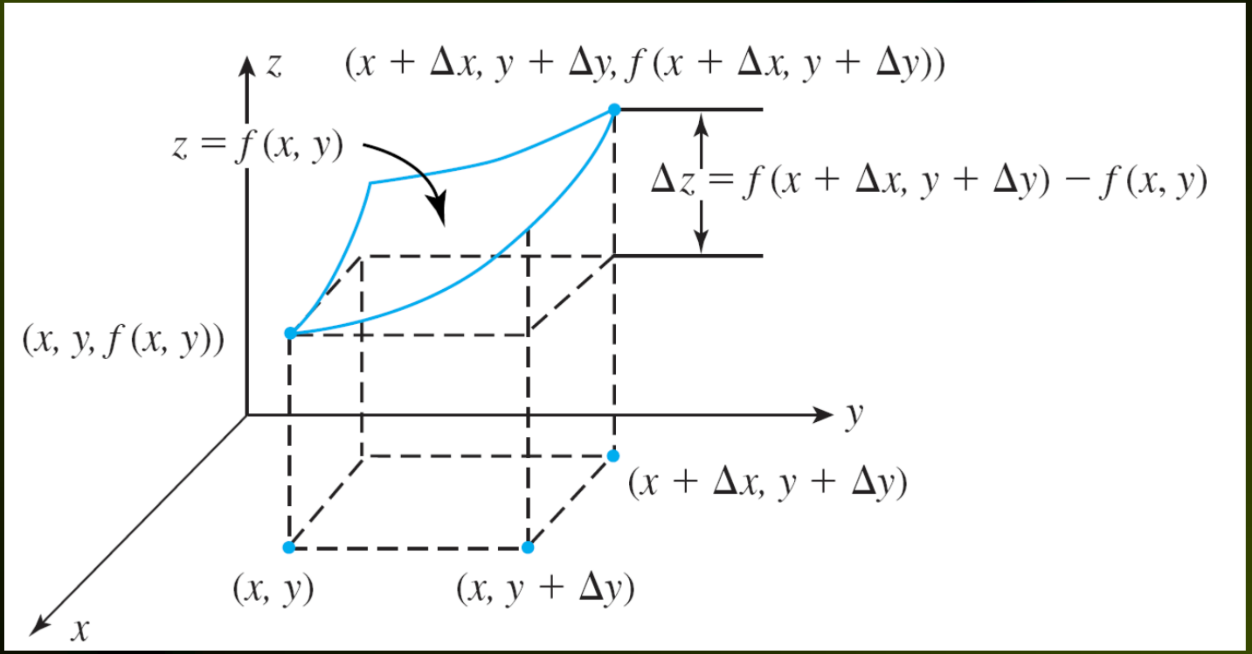
For example, if z is a function of two variables defined by $z = f(x, y)$, then the **increment** in z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) \qquad \textbf{(12)}$$

where Δx and Δy are the increments in the independent variables x and y, respectively.

Increments

See Figure 30b.



(b) The increment Δz is the change in z as x changes from x to $x + \Delta x$ and y changes from y to $y + \Delta y$.

Figure 30

Example 1

Let z = $f(x + y) = 2x^2 - xy$. Find Δz. Then use your result to find the change in z if (x, y) changes from (1, 1) to (0.98, 1.03).

Solution:

Using (12), we obtain

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

$$= [2(x + \Delta x)^2 - (x + \Delta x)(y + \Delta y)] - (2x^2 - xy)$$

$$= 2x^2 + 4x\Delta x + 2(\Delta x)^2 - xy - x\Delta y - y\Delta x - \Delta x\Delta y - 2x^2 + xy$$

$$= (4x - y)\Delta x - x\Delta y + 2(\Delta x)^2 - \Delta x\Delta y$$

Example 1 – *Solution*

cont'd

Next, to find the increment in z if (x, y) changes from (1, 1) to (0.98, 1.03), we note that

$$\Delta x = 0.98 - 1 = -0.02 \quad \text{and} \quad \Delta y = 1.03 - 1 = 0.03.$$

Therefore, using the result obtained earlier with $x = 1$, $y = 1$, $\Delta x = -0.02$, and $\Delta y = 0.03$, we obtain

$$\Delta z = [(4(1) - 1](-0.02) - (1)(0.03) + 2(-0.02)^2 - (-0.02)(0.03)$$

$$= -0.0886$$

You can verify the correctness of this result by calculating the quantity $f(0.98, 1.03) - f(1, 1)$.

The Total Differential

The Total Differential

Recall that if f is a function of one variable defined by $y = f(x)$, then the differential of f at x is defined by

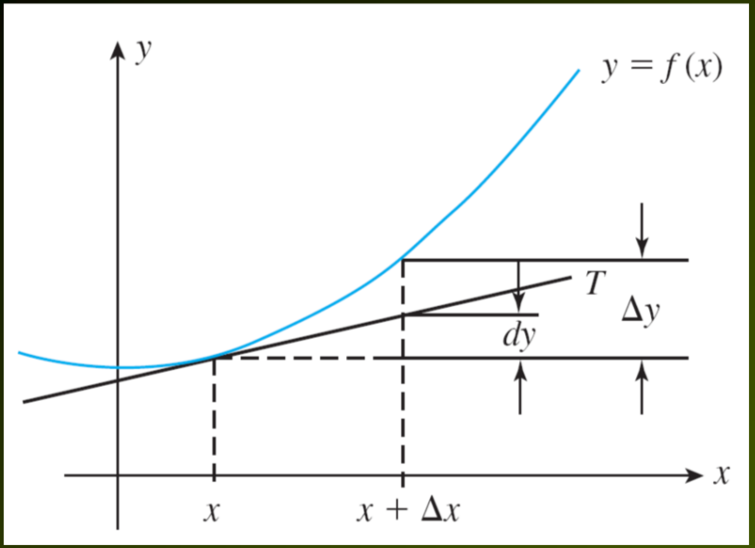
$$dy = f'(x)dx$$

where $dx = \Delta x$ is the differential in x.

Furthermore, we saw that

$$\Delta y \approx dy$$

if Δx is small (Figure 31a).



(a) Relationship between dy and Δy

Figure 31

The Total Differential

The concept of the differential extends readily to a function of two or more variables.

Total Differential

Let $z = f(x, y)$ define a differentiable function of x and y.

1. The **differentials** of the independent variables x and y are $dx = \Delta x$ and $dy = \Delta y$.
2. The **differential** of the dependent variable z is

$$dz = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy \qquad \textbf{(13)}$$

The Total Differential

Thus, analogous to the one-variable case, the total differential of z is a linear function of dx and dy.

Furthermore, it provides us with an approximation of the exact change in z,

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

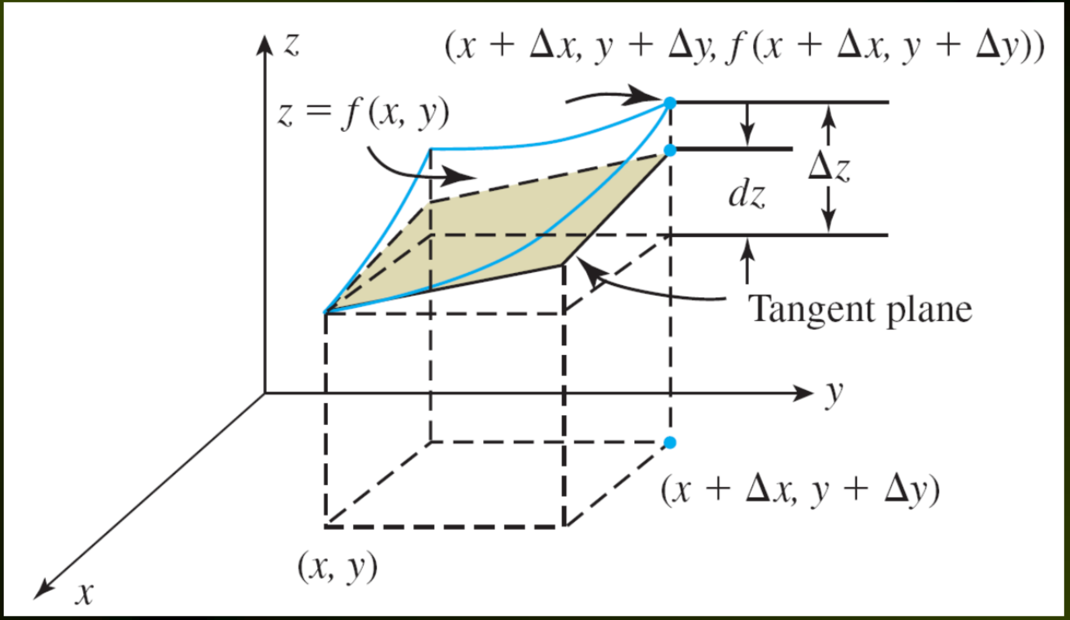
corresponding to a net change Δx in x from x to $x + \Delta x$ and a net change Δy in y from y to $y + \Delta y$; that is,

$$\Delta z \approx dz = \frac{\partial f}{\partial x}(x, y)\, dx + \frac{\partial f}{\partial y}(x, y)\, dy \qquad \textbf{(14)}$$

provided $\Delta x = dx$ and $\Delta y = dy$ are sufficiently small.

The Total Differential

See figure 31b.



(b) Relationship between *dz* and Δ*z*. The tangent plane is the analog of tangent line *T* in the one-variable case.

Figure 31

Example 2

Let $z = 2x^2y + y^3$.

a. Find the differential dz of z.

b. Find the approximate change in z when x changes from $x = 1$ to $x = 1.01$ and y changes from $y = 2$ to $y = 1.98$.

c. Find the actual change in z when x changes from $x = 1$ to $x = 1.01$ and y changes from $y = 2$ to $y = 1.98$. Compare the result with that obtained in part (b).

Example 2 – *Solution*

a. Let $f(x, y) = 2x^2y + y^3$.

Then the required differential is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= 4xy\,dx + (2x^2 + 3y^2)\,dy$$

b. Here $x = 1$, $y = 2$, and $dx = 1.01 - 1 = 0.01$ and $dy = 1.98 - 2 = -0.02$.

Therefore,

$$\Delta z \approx dz = 4(1)(2)(0.01) + [2(1) + 3(4)](-0.02)$$

$$= -0.20$$

Example 2 – *Solution*

cont'd

c. The actual change in z is given by

$$\Delta z = f(1.01, 1.98) - f(1, 2)$$

$$= [2(1.01)^2(1.98) + (1.98)^3] - [2(1)^2(2) + (2)^3]$$

$$\approx 11.801988 - 12$$

$$\approx -0.1980$$

We see that $\Delta z \approx dz$, as expected.

The Total Differential

If f is a function of the three variables x, y, and z, then the total differential of $w = f(x, y, z)$ is defined to be

$$dw = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

where $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ are the actual changes in the independent variables x, y, and z as x changes from $x = a$ to $x = a + \Delta x$, y changes from $y = b$ to $y = b + \Delta y$, and z changes from $z = c$ to $z = c + \Delta z$, respectively.