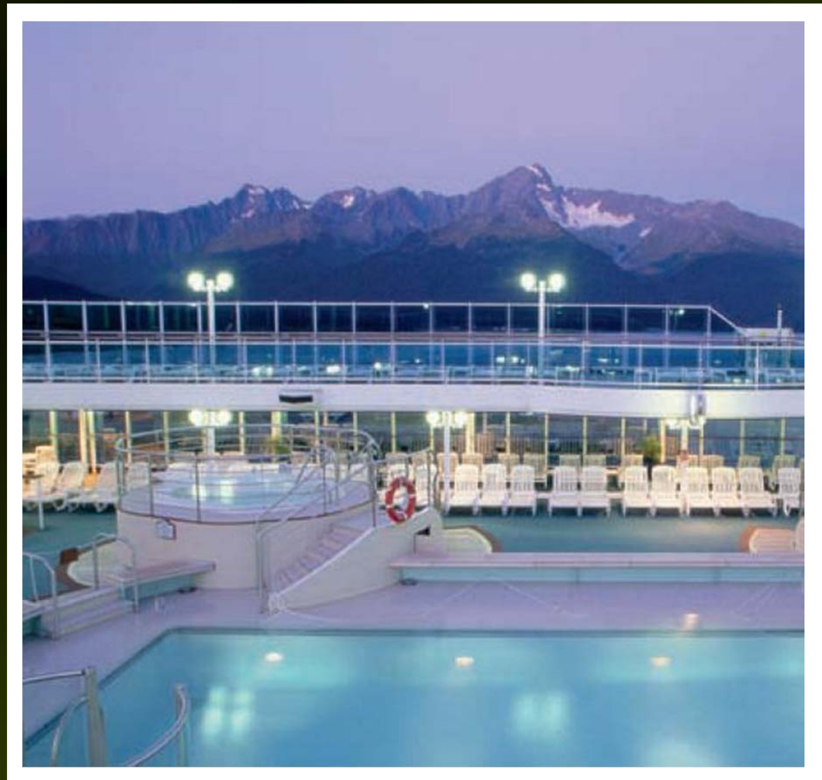
8

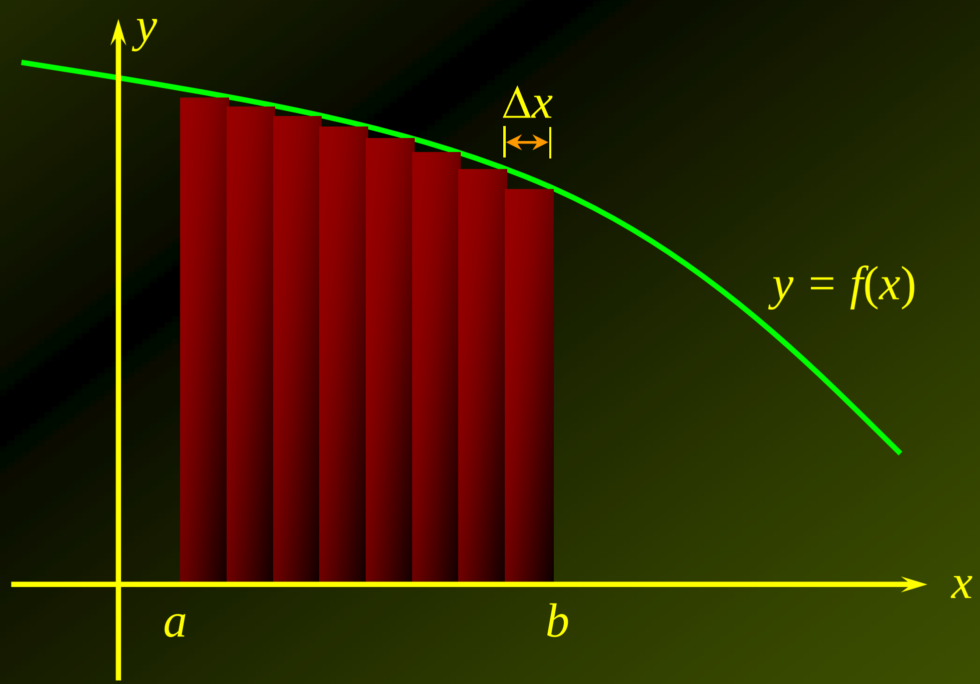
CALCULUS OF SEVERAL VARIABLES



8.7 Double Integrals

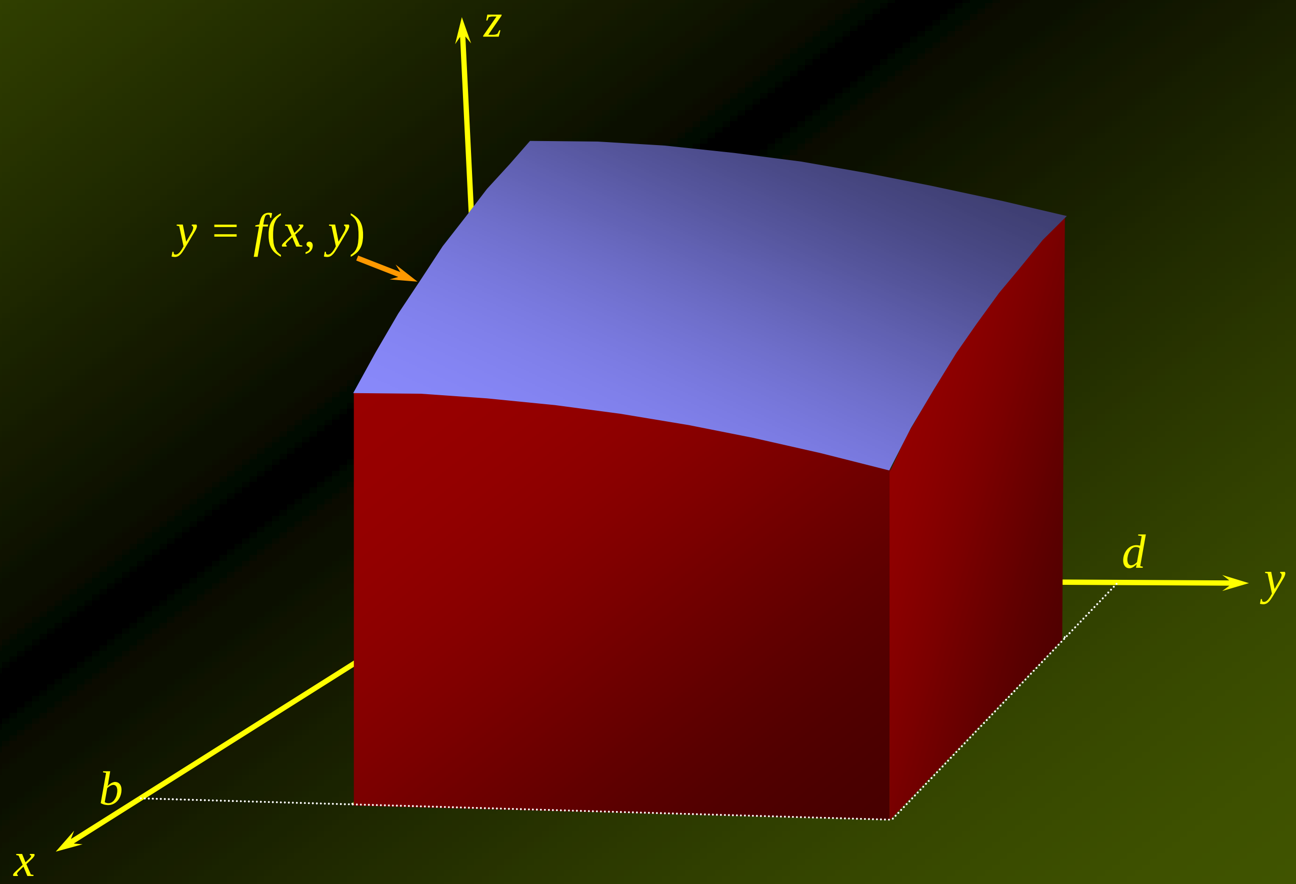
A Geometric Interpretation of the Double Integral

You may recall that we can do a Riemann sum to approximate the area under the graph of a function of one variable by adding the areas of the rectangles that form below the graph resulting from small increments of x (Δx) within a given interval $[a, b]$:



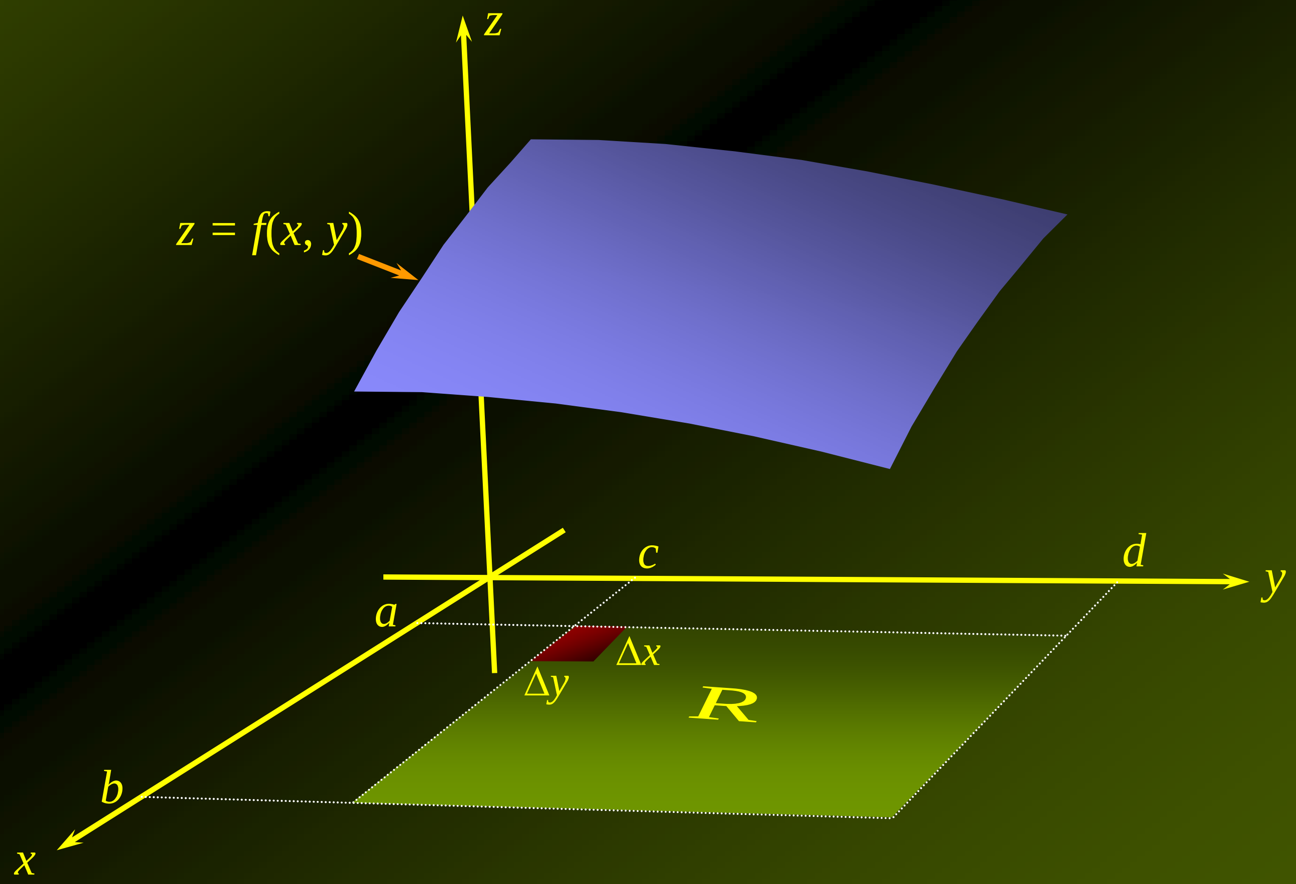
A Geometric Interpretation of the Double Integral

Similarly, it is possible to obtain an approximation of the volume of the solid under the graph of a function of two variables.



A Geometric Interpretation of the Double Integral

To find the volume of the solid under the surface, we can perform a Riemann sum of the volume S_i of parallelepipeds with base $R_i = \Delta x \times \Delta y$ and height $f(x_i, y_i)$:



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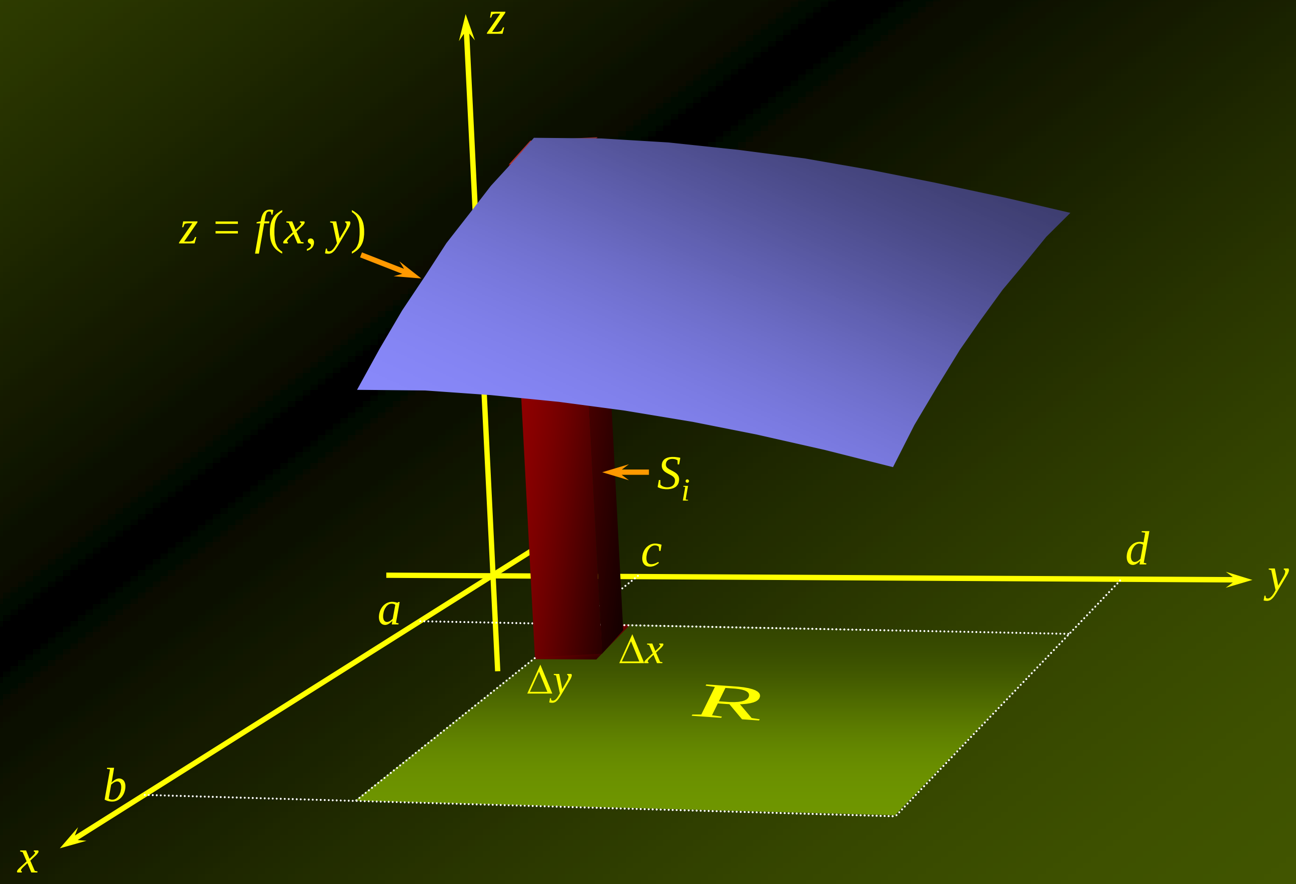

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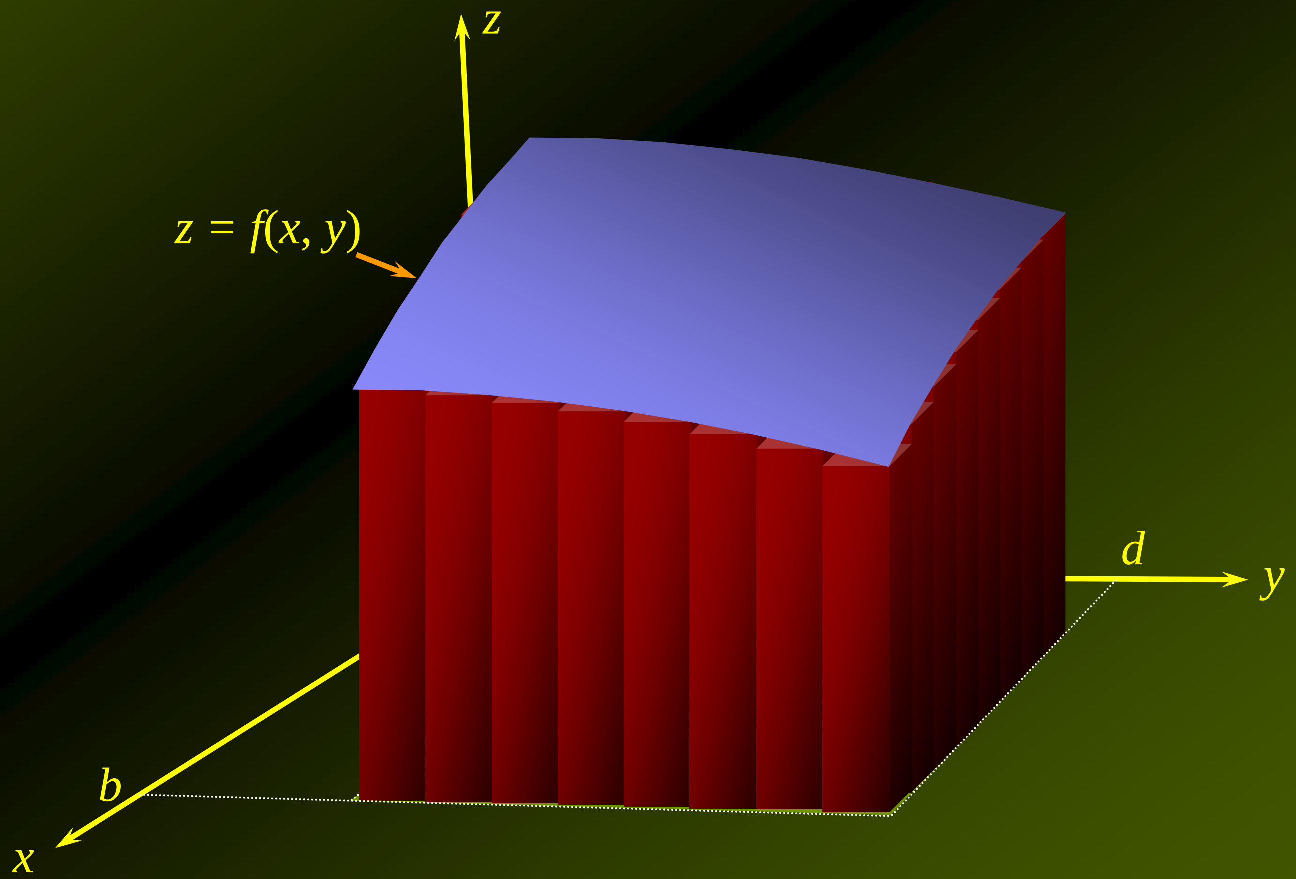
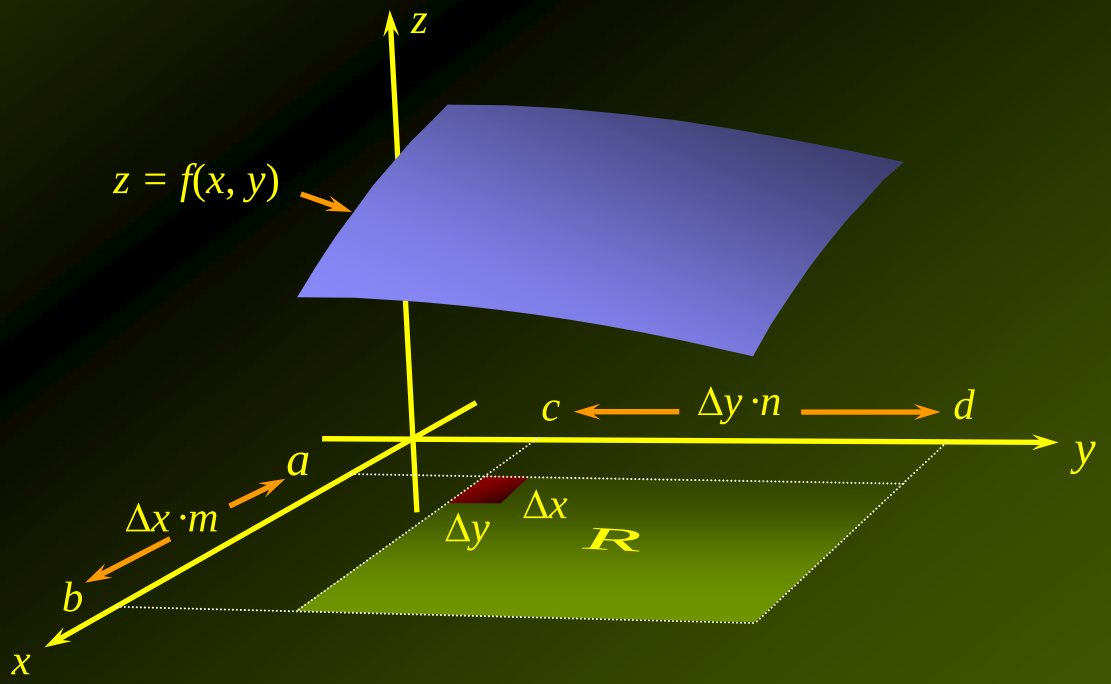


A Geometric Interpretation of the Double Integral

The limit of the Riemann sum obtained when the number of rectangles *m* along the *x*-axis, and the number of subdivisions *n* along the *y*-axis tends to infinity is the value of the double integral of *f*(*x*, *y*) over the region *R* and is denoted by $\int_R\int f(x,y)dA$



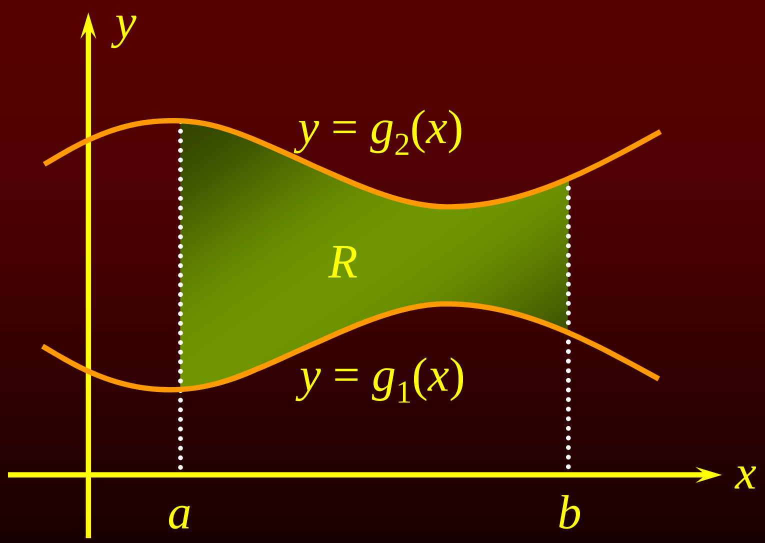
Theorem 1: Evaluating a Double Integral Over a Plane Region

a. Suppose $g_1(x)$ and $g_2(x)$ are continuous functions on $[a, b]$ and the region R is defined by

$R = \{(x, y) | \ g_1(x) \le y \le g_2(x);\ a \le x \le b\}$.

Then,

$$\int_R\int f(x,y)\,dA = \int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right]dx$$



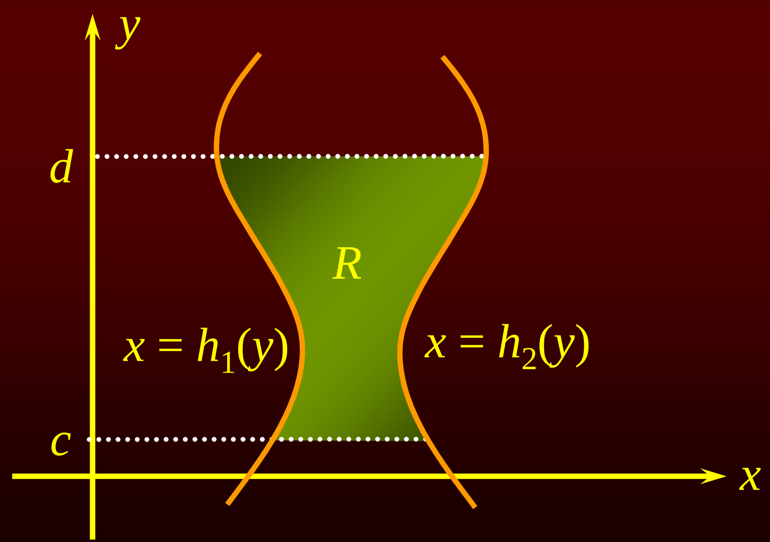
Theorem 1: Evaluating a Double Integral Over a Plane Region

b. Suppose $h_1(y)$ and $h_2(y)$ are continuous functions on $[c, d]$ and the region R is defined by

$R = \{(x, y) | h_1(y) \le x \le h_2(y); c \le y \le d\}$.

Then,

$$\int_R \int f(x, y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\,dx \right] dy$$

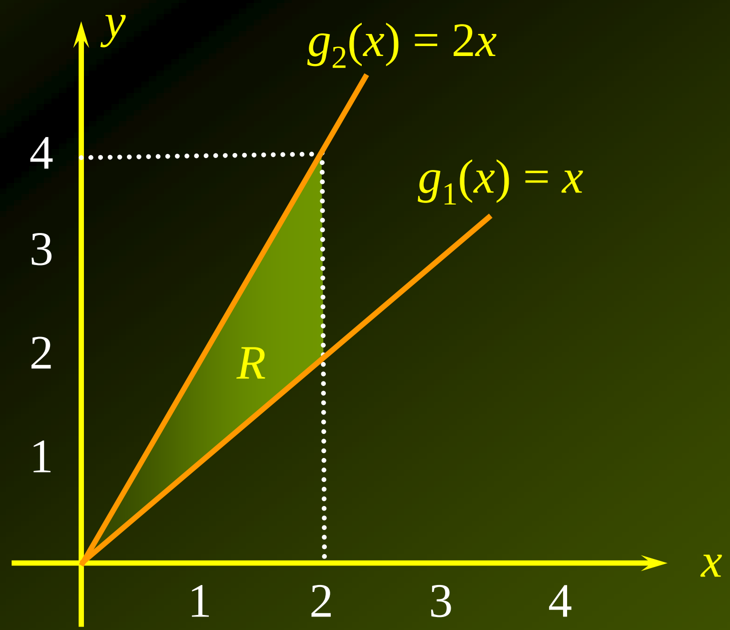


Example 2

Evaluate $\int_R\int f(x, y)dA$ given that $f(x, y) = x^2 + y^2$ and R is the region bounded by the graphs of $g_1(x) = x$ and $g_2(x) = 2x$ for $0 \leq x \leq 2$.

Solution:

The region under consideration is:



Example 2 – *Solution*

cont'd

Using *Theorem 1*, we find:

$$\int_R\int f(x,y)\,dA = \int_0^2\left[\int_x^{2x}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^2\left[\left(x^2y+\frac{1}{3}y^3\right)\Bigg|_x^{2x}\right]dx$$

$$= \int_0^2\left[\left(2x^3+\frac{8}{3}x^3\right)-\left(x^3+\frac{1}{3}x^3\right)\right]dx$$

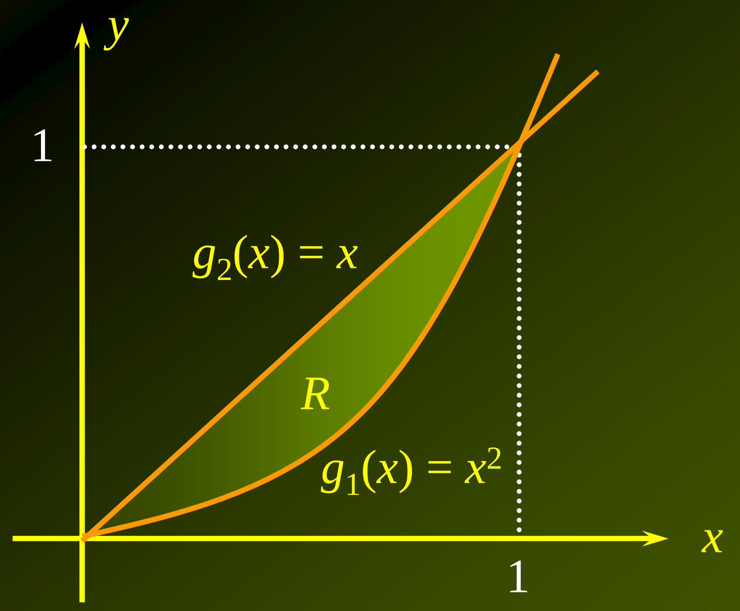
$$= \int_0^2\frac{10}{3}x^3\,dx \quad = \frac{5}{6}x^4\Bigg|_0^2 \quad = 13\tfrac{1}{3}$$

Example 3

Evaluate $\int_R\int f(x, y)\, dA$, where $f(x, y) = xe^y$ and R is the plane region bounded by the graphs of $y = x^2$ and $y = x$.

Solution:

The region under consideration is:



The points of intersection of the two curves are found by solving the equation $x^2 = x$, giving $x = 0$ and $x = 1$.

Example 3 – *Solution*

cont'd

Using *Theorem 1*, we find:

$$\int_R\int f(x,y)\,dA = \int_0^1\left[\int_{x^2}^{x} xe^y\,dy\right]dx = \int_0^1\left[\left(xe^y\right)\Big|_{x^2}^{x}\right]dx$$

$$= \int_0^1 (xe^x - xe^{x^2})\,dx = \int_0^1 xe^x\,dx - \int_0^1 xe^{x^2}\,dx$$

$$= \left[(x-1)e^x - \frac{1}{2}e^{x^2}\right]\Bigg|_0^1$$

Integrating by parts on the right-hand side

$$= -\frac{1}{2}e - \left(-1 - \frac{1}{2}\right) = \frac{1}{2}(3-e)$$

The Volume of a Solid Under a Surface

Let R be a region in the xy-plane and let f be continuous and nonnegative on R.

Then, the volume of the solid under a surface bounded above by $z = f(x, y)$ and below by R is given by

$$V = \iint_R f(x, y)\, dA$$

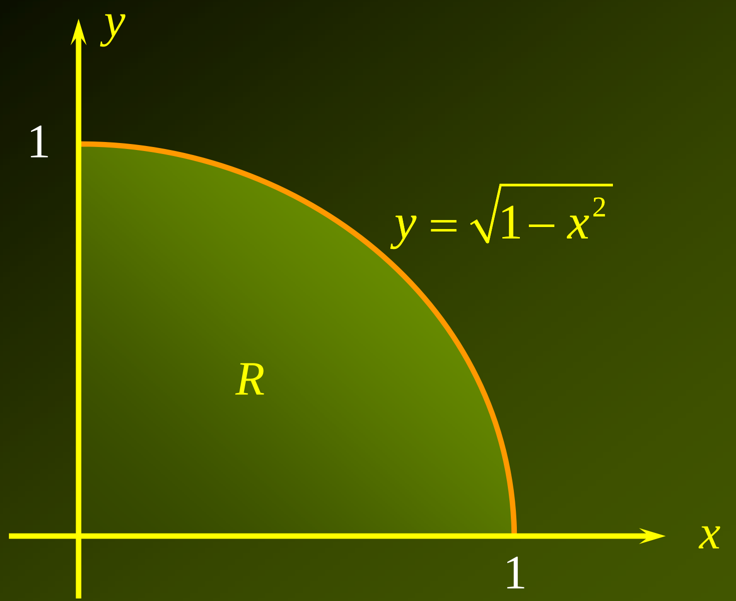
Example 4

Find the volume of the solid bounded above by the plane $z = f(x, y) = y$ and below by the plane region R defined by

$$y = \sqrt{1 - x^2} \qquad (0 \le x \le 1)$$

Solution:

The graph of the region R is:



Observe that $f(x, y) = y > 0$ for (x, y) in R.

Example 4 – *Solution*

cont'd

Therefore, the required volume is given by

$$V = \int_R \int y\,dA = \int_0^1 \left[\int_0^{\sqrt{1-x^2}} y\,dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{2} y^2 \Big|_0^{\sqrt{1-x^2}} \right] dx$$

$$= \int_0^1 \frac{1}{2}(1 - x^2)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{3}x^3 \right)\Big|_0^1 = \frac{1}{3}$$

Example 4 – *Solution*

cont'd

The graph of the solid in question is: