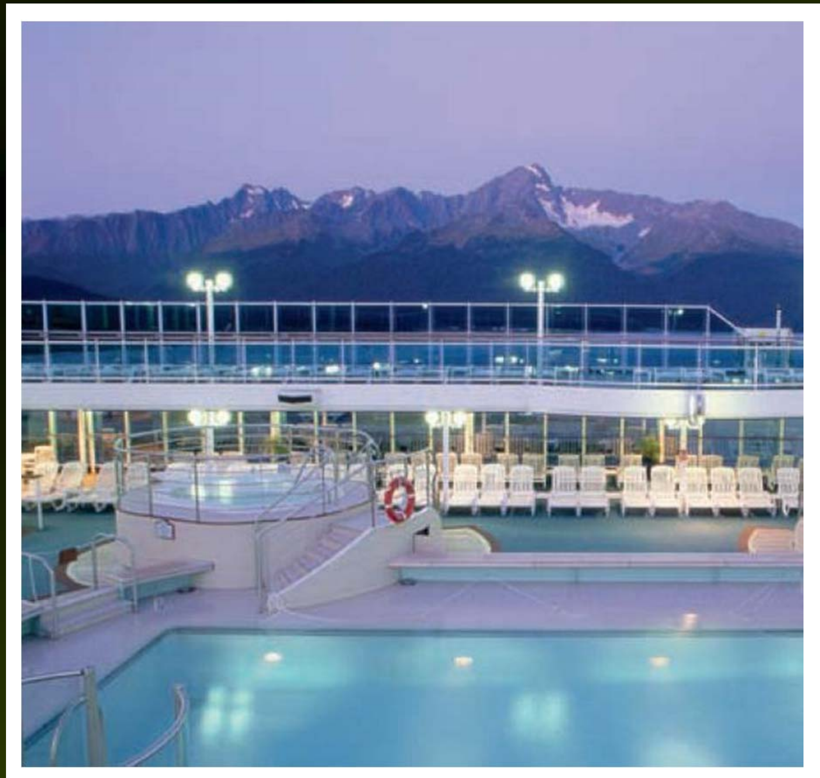
8

CALCULUS OF SEVERAL VARIABLES



8.8 Applications of Double Integrals

Finding the Volume of a Solid by Double Integrals

Finding the Volume of a Solid by Double Integrals

As we saw earlier, the double integral

$$\iint_R f(x, y)\, dA$$

gives the volume of the solid bounded by the graph of *f*(*x*, *y*) over the region *R*.

The Volume of a Solid under a Surface

Let R be a region in the xy-plane and let f be continuous and nonnegative on R. Then, the **volume of the solid under a surface** bounded above by $z = f(x, y)$ and below by R is given by

$$V = \iint_R f(x, y)\, dA$$

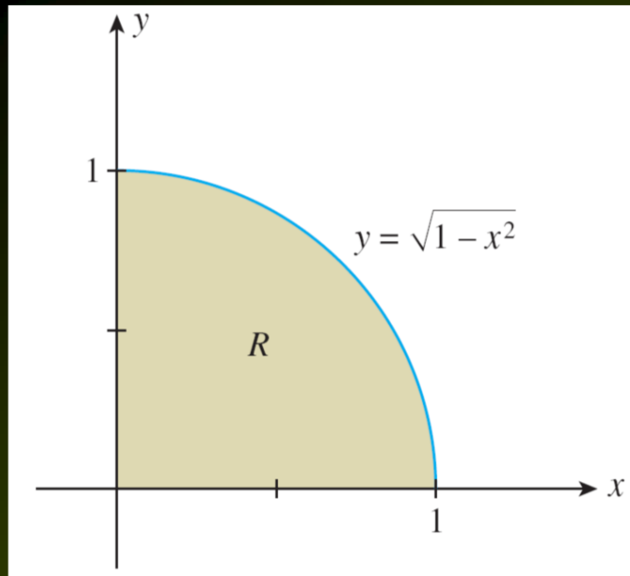
Example 1

Find the volume of the solid bounded above by the plane $z = f(x, y) = y$ and below by the plane region R defined by $y = \sqrt{1 ? x^2}$ $(0 \leq x \leq 1)$.

Solution:

The region R is sketched in Figure 40.

Observe that $f(x, y) = y \geq 0$ for $(x, y) \in R$.



The plane region R defined by $y = \sqrt{1 ? x^2}$ $(0 \leq x \leq 1)$

Figure 40

Example 1 – *Solution*

Therefore, the required volume is given by

$$\iint_R y \; dA = \int_0^1 \left[\int_0^{\sqrt{1 ? x^2}} y \; dy \right] dx = \int_0^1 \left[\frac{1}{2} y^2 \Big|_0^{\sqrt{1 ? x^2}} \right] dx$$

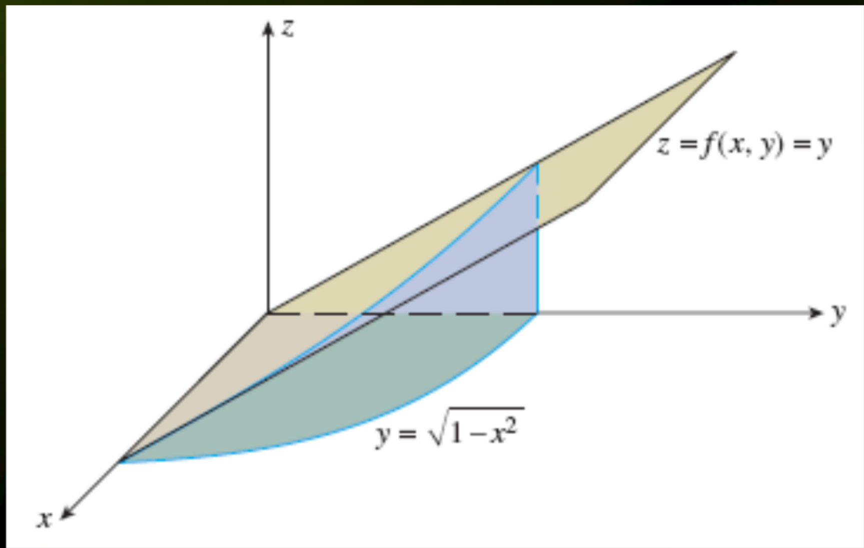
$$= \int_0^1 \frac{1}{2} (1 ? x^2 \;\; dx = \frac{1}{2} \left(x ? \frac{1}{3} x^3 \right) \Big|_0^1 = \frac{1}{3}$$

or $\frac{1}{3}$ cubic unit.

Example 1 – *Solution*

cont'd

The solid is shown in Figure 41. Note that it is not necessary to make a sketch of the solid in order to compute its volume.



The solid bounded above by the plane $z = y$ and below by the plane region defined by $y = \sqrt{1 - x^2}$ $(0 \leq x \leq 1)$

Figure 41

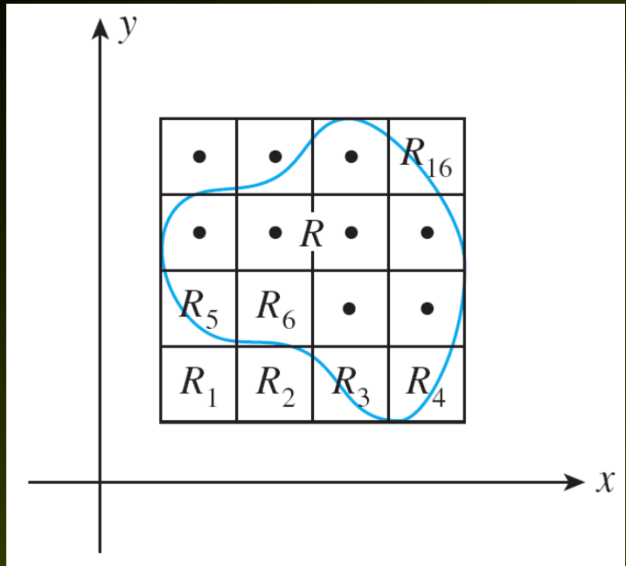
Population of a City

Population of a City

Suppose the plane region R represents a certain district of a city and $f(x, y)$ gives the population density (the number of people per square mile) at any point (x, y) in R.

Enclose the set R by a rectangle and construct a grid for it in the usual manner.

In any rectangular region of the grid that has no point in common with R, set $f(x_i, y_i)hk = 0$ (Figure 42).



The rectangular region R representing a certain district of a city is enclosed by a rectangular grid.

Figure 42

Population of a City

Then, corresponding to any grid covering the set R, the general term of the Riemann sum $f(x_i, y_i)hk$ (population density times area) gives the number of people living in that part of the city corresponding to the rectangular region R_i.

Therefore, the Riemann sum gives an approximation of the number of people living in the district represented by R and, in the limit, the double integral

$$\iint_R f(x, y)\, dA$$

gives the actual number of people living in the district under consideration.

Applied Example 2 – *Population Density*

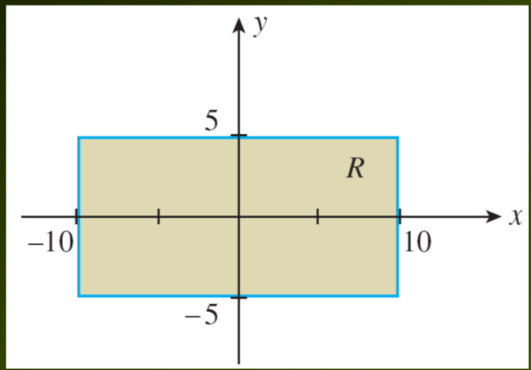
The population density of a certain city is described by the function

$$f(x, y) = 10{,}000e^{-0.2|x|-0.1|y|}$$

where the origin (0, 0) gives the location of the city hall. What is the population inside the rectangular area described by

$$R = \{(x, y) \mid -10 \le x \le 10; -5 \le y \le 5\}$$

if x and y are in miles? (See Figure 43.)



The rectangular region R represents a certain district of a city.

Figure 43

Applied Example 2 – *Solution*

By symmetry, it suffices to compute the population in the first quadrant. (Why?) Then, upon observing that in this quadrant

$$f(x, y) = 10{,}000e^{-0.2x-0.1y} = 10{,}000e^{-0.2x}e^{-0.1y}$$

we see that the population in R is given by

$$\iint_R f(x, y)\, dA = 4\int_0^{10}\left[\int_0^{5} 10{,}000e^{?\ .2x}e^{?\ .1y}\, dy\right] dx$$

$$= 4\int_0^{10}\left[?\ \ 00{,}000e^{?\ .2x}e^{?\ .1y}\Big|_0^5\right] dx$$

$$= 400{,}000(1 - e^{-0.5})\int_0^{10} e^{-0.2x}\, dx$$

Applied Example 2 – *Solution*

cont'd

$$= 2{,}000{,}000(1 - e^{-0.5})(1 - e^{-2})$$

or approximately 680,438.

Average Value of a Function

Average Value of a Function

We know that the average value of a continuous function $f(x)$ over an interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

That is, the average value of a function over $[a, b]$ is the integral of f over $[a, b]$ divided by the length of the interval. An analogous result holds for a function of two variables $f(x, y)$ over a plane region R.

To see this, we enclose R by a rectangle and construct a rectangular grid. Let (x_i, y_i) be any point in the rectangle R_i of area hk.

Average Value of a Function

Now, the average value of the mn numbers $f(x_1, y_1)$, $f(x_2, y_2), \ldots, f(x_{mn}, y_{mn})$ is given by

$$\frac{f(x_1, y_1) + f(x_2, y_2) + \cdots + f(x_{mn}, y_{mn})}{mn}$$

which can also be written as

$$\frac{hk}{hk}\left[\frac{f(x_1, y_1) + f(x_2, y_2) + \cdots + f(x_{mn}, y_{mn})}{mn}\right]$$

$$= \frac{1}{(mn)hk}\left[f(x_1, y_1) + f(x_2, y_2) + \cdots + f(x_{mn}, y_{mn})\right]hk$$

Average Value of a Function

Now the area of *R* is approximated by the sum of the *mn* rectangles (*omitting* those having no points in common with *R*), each of area *hk*.

Note that this is the denominator of the previous expression. Therefore, taking the limit as *m* and *n* both tend to infinity, we obtain the following formula for the *average value of f*(*x*, *y*) *over R.*

Average Value of $f(x, y)$ over the Region R

If f is integrable over the plane region R, then its average value over R is given by

$$\frac{\iint_R f(x, y)\, dA}{\text{Area of } R} \quad \text{or} \quad \frac{\iint_R f(x, y)\, dA}{\iint_R dA} \qquad \textbf{(18)}$$

Average Value of a Function

Note:

If we let $f(x, y) = 1$ for all (x, y) in R, then

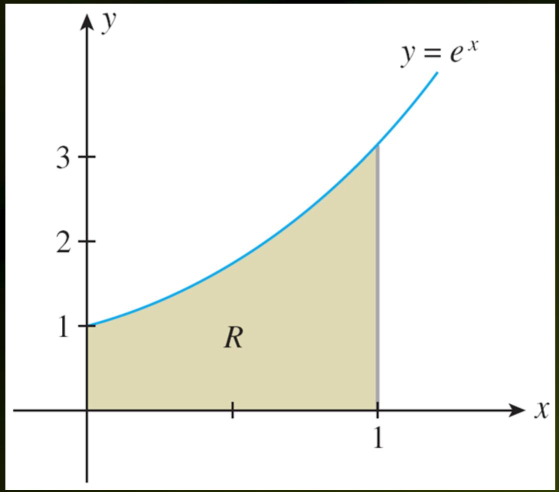
$$\iint_R f(x, y)\, dA = \iint_R dA = \text{Area of } R$$

Example 3

Find the average value of the function $f(x, y) = xy$ over the plane region defined by $y = e^x$ $(0 \leq x \leq 1)$.

Solution:

The region R is shown in Figure 44.



The plane region R defined by $y = e^x$ $(0 \leq x \leq 1)$

Figure 44

Example 3 – *Solution*

cont'd

The area of the region *R* is given by

$$\int_0^1 \left[\int_0^{e^x} dy \right] dx = \int_0^1 \left[y \Big|_0^{e^x} \right] dx$$

$$= \int_0^1 e^x \, dx$$

$$= e^x \Big|_0^1$$

$$= e - 1$$

We would obtain the same result had we viewed the area of this region as the area of the region under the curve $y = e^x$ from $x = 0$ to $x = 1$.

Example 3 – *Solution*

cont'd

Next, we compute

$$\iint_R f(x, y)\, dA = \int_0^1 \left[\int_0^{e^x} xy\, dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{2} xy^2 \Big|_0^{e^x} \right] dx$$

$$= \int_0^1 \frac{1}{2} xe^{2x}\, dx$$

$$= \frac{1}{4} xe^{2x} \; ? \; \frac{1}{8} e^{2x} \Big|_0^1$$

Integrate by parts.

Example 3 – *Solution*

cont'd

$$= \left(\frac{1}{4}e^2 \; ? \; \frac{1}{8}e^2 \right) \quad \frac{1}{8}$$

$$= \frac{1}{8}(e^2 + 1)$$

Therefore, the required average value is given by

$$\frac{\iint_R f(x, y)\, dA}{\iint_R dA} = \frac{\frac{1}{8}(e^2 + 1)}{e \; ?} = \frac{e^2 + 1}{8(e \; ? \quad)}$$