9 DIFFERENTIAL EQUATIONS

9.1 Differential Equations

Models Involving Differential Equations

Models Involving Differential Equations

Recall that a differential equation is an equation that involves an unknown function and its derivative(s). Here are some examples of differential equations:

$$\frac{dy}{dx} = xe^x \qquad \frac{dy}{dx} + 2y = x^2 \qquad \frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + ty\ ? \quad = 0$$

Differential equations appear in practically every branch of applied mathematics, and the study of these equations remains one of the most active areas of research in mathematics.

As you will see in the next few examples, models involving differential equations often arise from the mathematical formulation of practical problems.

Models Involving Differential Equations

Unrestricted Growth Models

We have seen that the size of a population at any time t, $Q(t)$, increases at a rate that is proportional to $Q(t)$ itself.

Thus,

$$\frac{dQ}{dt} = kQ \qquad \textbf{(1)}$$

where k is a constant of proportionality.

This is a differential equation involving the unknown function Q and its derivative Q'.

Models Involving Differential Equations

Restricted Growth Models

In many applications the quantity $Q(t)$ does not exhibit unrestricted growth but approaches some definite upper bound.

The learning curves and logistic functions are examples of restricted growth models.

Let’s derive the mathematical models that lead to these functions.

Models Involving Differential Equations

Suppose $Q(t)$ does not exceed some number C, called the *carrying capacity of the environment*.

Furthermore, suppose the rate of growth of this quantity is *proportional* to the difference between its upper bound and its current size.

The resulting differential equation is

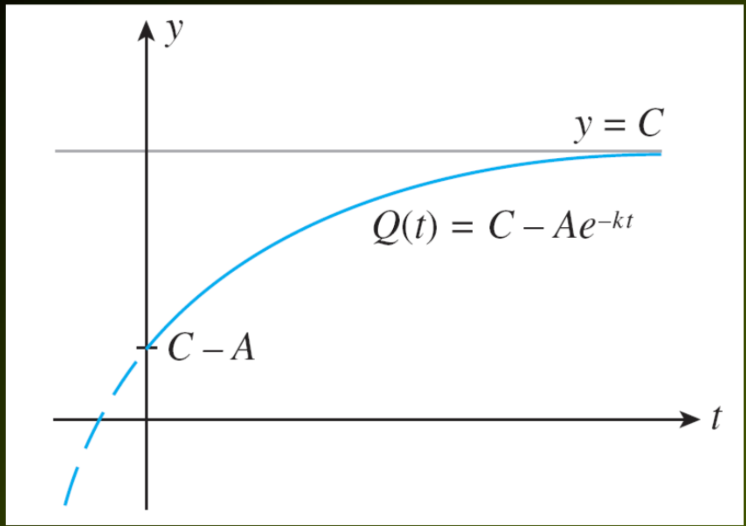
$$\frac{dQ}{dt} = K(C-Q) \qquad \textbf{(2)}$$

where k is a constant of proportionality.

Models Involving Differential Equations

Observe that if the initial population is small relative to *C*, then the rate of growth of *Q* is relatively large. But as *Q*(*t*) approaches *C*, the difference *C* – *Q*(*t*) approaches zero, as does the rate of growth of *Q*.

The solution of the differential Equation (2) is a function that describes a learning curve (Figure 1).



Q(*t*) describes a learning curve.

Figure 1

Models Involving Differential Equations

Next, let's consider a restricted growth model in which the rate of growth of a quantity $Q(t)$ is *jointly proportional* to its current size and the difference between its upper bound and its current size; that is,

$$\frac{dQ}{dt} = \boldsymbol{kQ(C-Q)} \qquad \textbf{(3)}$$

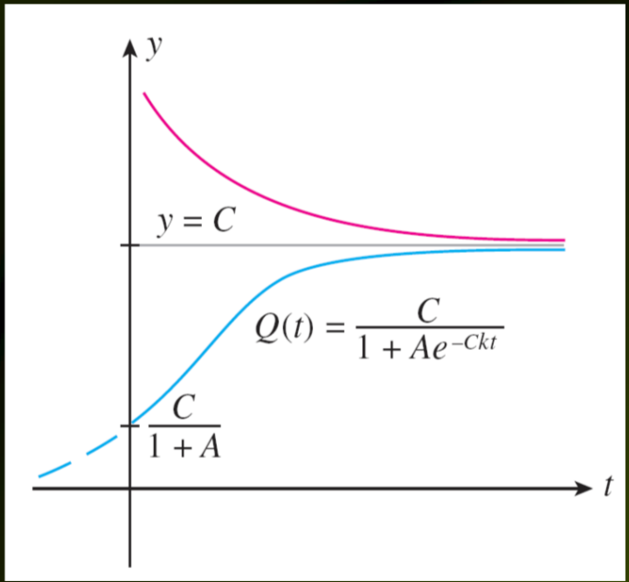
where k is a constant of proportionality.

Observe that when $Q(t)$ is small relative to C, the rate of growth of Q is approximately proportional to Q. But as $Q(t)$ approaches C, the growth rate slows down to zero.

Models Involving Differential Equations

If $Q > C$, then $dQ/dt < 0$ and the quantity is decreasing with time, with the decay rate slowing down as Q approaches C.

The solution of the differential Equation (3) is just the logistic function. Its graph is shown in Figure 2.



Two solutions of the logistic equation

Figure 2

Models Involving Differential Equations

Stimulus Response

In the quantitative theory of psychology, one model that describes the relationship between a stimulus *S* and the resulting response *R* is the Weber–Fechner law.

This law asserts that the rate of change of a reaction *R* is inversely proportional to the stimulus *S*. Mathematically, this law may be expressed as

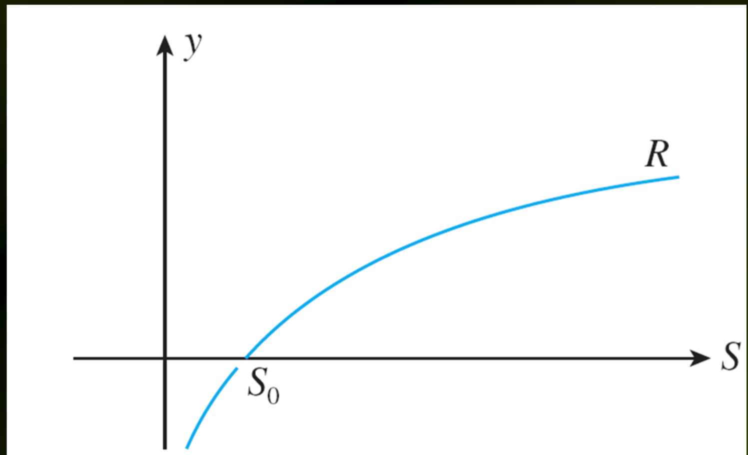
$$\frac{dR}{dS} = \frac{k}{S} \qquad \textbf{(4)}$$

where *k* is a constant of proportionality.

Models Involving Differential Equations

Furthermore, suppose that the threshold level, the lowest level of stimulation at which sensation is detected, is S_0. Then we have the condition $R = 0$ when $S = S_0$; that is, $R(S_0) = 0$.

The graph of R versus S is shown in Figure 3.



The solution to the differential equation (4) describes the response to a stimulus.

Figure 3

Models Involving Differential Equations

Mixture Problems

Our next example is a typical mixture problem. Suppose a tank initially contains 10 gallons of pure water. Brine containing 3 pounds of salt per gallon flows into the tank at a rate of 2 gallons per minute, and the well-stirred mixture flows out of the tank at the same rate. How much salt is in the tank at any given time?

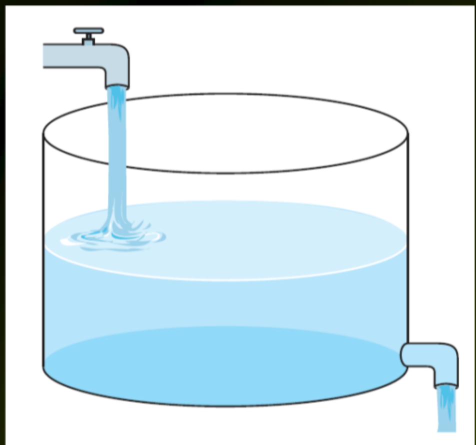
Let's formulate this problem mathematically. Suppose $A(t)$ denotes the amount of salt in the tank at any time t.

Models Involving Differential Equations

Then the derivative *dA*/*dt*, the rate of change of the amount of salt at any time *t*, must satisfy the condition

$$\frac{dA}{dt} = \text{(Rate of salt flowing in)} - \text{(Rate of salt flowing out)}$$

(Figure 4).



The rate of change of the amount of salt at time *t* = (Rate of salt flowing in) – (Rate of salt flowing out)

Figure 4

Models Involving Differential Equations

But the rate at which salt flows into the tank is given by

(2 gal/min)(3 lb/gal)   (Rate of flow) × (Concentration)

or 6 pounds per minute.

Since the rate at which the solution leaves the tank is the same as the rate at which the brine is poured into it, the tank contains 10 gallons of the mixture at any time t.

Since the salt content at any time t is A pounds, the concentration of the salt in the mixture is (A/10) pounds per gallon.

Models Involving Differential Equations

Therefore, the rate at which salt flows out of the tank is given by

$$(2 \text{ gal/min})\left(\frac{A}{10} \text{ lb/gal}\right)$$

or *A*/5 pounds per minute. Therefore, we are led to the differential equation

$$\frac{dA}{dt} = 6 \;?\; \frac{A}{5} \qquad \textbf{(5)}$$

An additional condition arises from the fact that initially there is no salt in the solution. This condition may be expressed mathematically as $A = 0$ when $t = 0$ or, more concisely, $A(0) = 0$.

Solutions of Differential Equations

Solutions of Differential Equations

Suppose we are given a differential equation involving the derivative(s) of a function y. Recall that a **solution** to a differential equation is any function $f(x)$ that satisfies the differential equation.

Thus, $y = f(x)$ is a solution of the differential equation provided that the replacement of y and its derivative(s) by the function $f(x)$ and its corresponding derivatives reduces the given differential equation to an identity for all values of x.

Example 2

Show that any function of the form $f(x) = ce^{-x} + x - 1$, where c is a constant, is a solution of the differential equation

$$y' + y = x$$

Solution:

Let

$$y = f(x) = ce^{-x} + x - 1$$

so that

$$y' = f'(x) = -ce^{-x} + 1$$

Example 2 – *Solution*

cont'd

Substituting these equations into the left side of the given differential equation yields

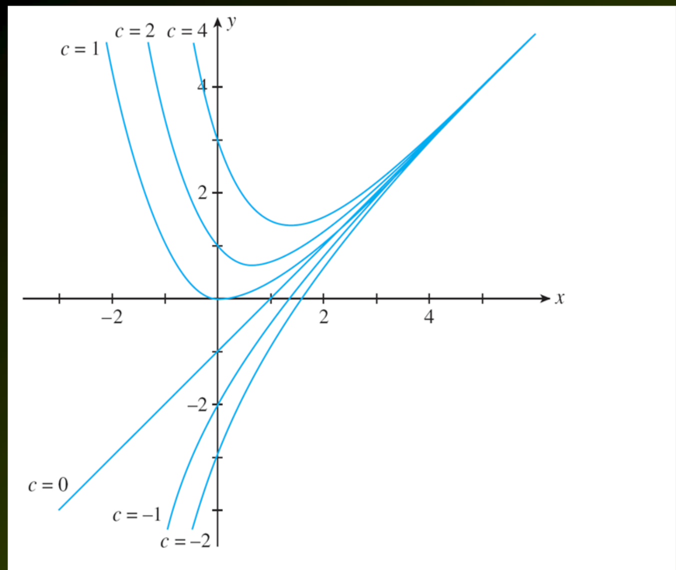
$$\overbrace{?ce^{?x} + 1}^{y'} + \overbrace{ce^{\ x} + x\ ?}^{y} \quad = x$$

and we have verified the assertion.

Solutions of Differential Equations

It can be shown that *every* solution of the differential equation $y' + y = x$ must have the form $y = ce^{-x} + x - 1$, where c is a constant; therefore, this is a general solution of the differential equation $y' + y = x$.

Figure 5 shows a family of solutions of this differential equation for selected values of c.



Some solutions of $y' + y = x$

Figure 5

Solutions of Differential Equations

Recall that the solution obtained by assigning a specific value to the constant c is called a particular solution of the differential equation.

For example, the particular solution $y = e^{-x} + x - 1$ is obtained from the general solution $y = ce^{-x} + x - 1$ by taking $c = 1$.

In practice, a particular solution of a differential equation is obtained from the general solution of the differential equation by requiring that the solution and/or its derivative(s) satisfy certain conditions at one or more values of x.