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DIFFERENTIAL EQUATIONS

9.2 Separation of Variables

The Method of Separation of Variables

The Method of Separation of Variables

Differential equations are classified according to their basic form. A compelling reason for this categorization is that different methods are used to solve different types of equations.

The **order** of a differential equation is the order of the highest derivative of the unknown function appearing in the equation.

A differential equation may be classified by its order.

The Method of Separation of Variables

For example, the differential equations

$$y' = xe^x \quad \text{and} \quad y' + 2y = x^2$$

are first-order equations, whereas the differential equation

$$\frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^3 + ty \ ? \ 8 = 0$$

is a second-order equation.

The Method of Separation of Variables

In this section we describe a method for solving an important class of first-order differential equations: those that can be written in the form

$$\frac{dy}{dx} = f(x)g(y)$$

where $f(x)$ is a function of x only and $g(y)$ is a function of y only. Such differential equations are said to be separable because the variables can be separated.

The Method of Separation of Variables

Following is the first-order separable differential equation.

$$\frac{dQ}{dt} = kQ(C - Q)$$

It has the form $dQ/dt = f(t)g(Q)$, where $f(t) = k$ and $g(Q) = Q(C - Q)$, and so it is separable. On the other hand, the differential equation

$$\frac{dy}{dx} = xy^2 + 2$$

is *not* separable.

The Method of Separation of Variables

Separable first-order equations can be solved using the method of separation of variables.

Method of Separation of Variables

Suppose we are given a first-order separable differential equation in the form

$$\frac{dy}{dx} = f(x)g(y) \qquad \textbf{(6)}$$

Step 1 Write Equation (6) in the form

$$\frac{dy}{g(y)} = f(x)\,dx \qquad \textbf{(7)}$$

When written in this form, the variables in (7) are said to be *separated*.

Step 2 Integrate each side of Equation (7) with respect to the appropriate variable.

Solving Separable Differential Equations

Example 1

Find the general solution of the first-order differential equation

$$y' = \frac{xy}{x^2 + 1}$$

Solution:

Step 1: Observe that the given differential equation has the form

$$\frac{dy}{dx} = \left(\frac{x}{x^2 + 1}\right)y = f(x)g(y)$$

where $f(x) = x/(x^2 + 1)$ and $g(y) = y$, and is therefore separable.

Example 1 – *Solution*

cont'd

Separating the variables, we obtain

$$\frac{dy}{y} = \left(\frac{x}{x^2 + 1}\right) dx$$

Step 2: Integrating each side of the last equation with respect to the appropriate variable, we have

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 1}\, dx$$

or

$$\ln |y| + C_1 = \frac{1}{2}\ln(x^2 + 1) + C_2$$

Example 1 – *Solution*

cont'd

$$\ln|y| = \frac{1}{2}\ln(x^2 + 1) + C_2 - C_1$$

where C_1 and C_2 are arbitrary constants of integration. If we choose C such that $C_2 - C_1 = \ln|C|$, then we have

$$\ln|y| = \frac{1}{2}\ln(x^2 + 1) + \ln|C|$$

$$= \ln\sqrt{x^2 + 1} + \ln|C|$$

$$= \ln|C\sqrt{x^2 + 1}|$$

$\ln A + \ln B = \ln AB$

so the general solution is

$$y = \sqrt{x^2 + 1}$$

Solving Separable Differential Equations

An initial value problem consists of a differential equation with one or more side conditions specified at a point.

Justification of the Method of Separation of Variables

Justification of the Method of Separation of Variables

To justify the method of separation of variables, let's consider the separable Equation (6) in its general form:

$$\frac{dy}{dx} = f(x)g(y)$$

If $g(y) \neq 0$, we may rewrite the equation in the form

$$\frac{1}{g(y)}\frac{dy}{dx} ? f(x) = 0$$

Justification of the Method of Separation of Variables

Now, suppose that G is an antiderivative of $1/g$ and F is an antiderivative of f. Using the chain rule, we see that

$$\frac{d}{dx}\left[G(y) ? F(x)\right] = G'(y)\frac{dy}{dx} ? F'(x)$$

$$= \frac{1}{g(y)}\frac{dy}{dx} ? f(x)$$

Therefore,

$$\frac{d}{dx}\left[G(y) ? F(x)\right] = 0$$

and so

$$G(y) - F(x) = C$$

C, a constant

Justification of the Method of Separation of Variables

But the last equation is equivalent to

$$G(y) = F(x) + C \quad \text{or} \quad \int \frac{dy}{g(y)} = \int f(x)\,dx$$

which is precisely the result of step 2 in the method of separation of variables.