9

DIFFERENTIAL EQUATIONS

9.3 Applications of Separable Differential Equations

Unrestricted Growth Models

Unrestricted Growth Models

The differential equation describing an unrestricted growth model is given by

$$\frac{dQ}{dt} = kQ$$

where $Q(t)$ represents the size of a certain population at time t and k is a positive constant.

Unrestricted Growth Models

Separating the variables in this differential equation and integrating, we have

$$\int \frac{dQ}{Q} = \int k\ dt$$

$$\ln|Q| = kt + C_1$$

$$|Q| = e^{kt + C_1} = C_2 e^{kt} \qquad C_2 = e^{C_1}$$

$$Q = Ce^{kt} \qquad C = \pm C_2$$

Thus, we may write the solution as

$$Q(t) = Ce^{kt}$$

Unrestricted Growth Models

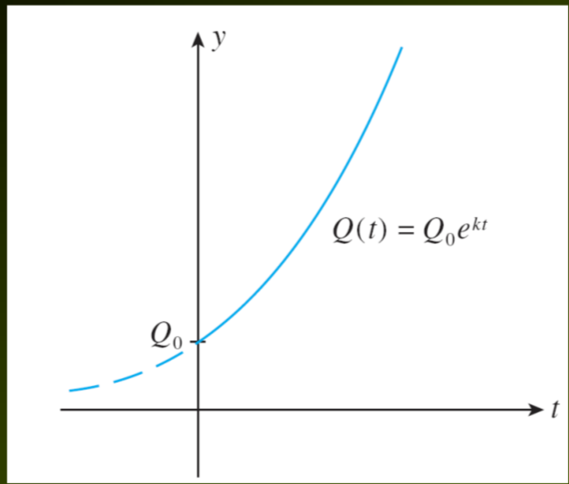
Observe that if the quantity present initially is denoted by Q_0, then $Q(0) = Q_0$. Applying this condition yields the equation

$$Ce^0 = Q_0 \qquad \text{or} \qquad C = Q_0$$

Therefore, the model for unrestricted exponential growth with initial population Q_0 is given by

$$Q(t) = Q_0e^{kt} \qquad \textbf{(8)}$$

(Figure 7).



An unrestricted growth model

Figure 7

Applied Example 1 – *Growth of Bacteria*

Under ideal laboratory conditions, the rate of growth of bacteria in a culture is proportional to the size of the culture at any time *t*. Suppose that 10,000 bacteria are present initially in a culture and 60,000 are present 2 hours later. How many bacteria will there be in the culture at the end of 4 hours?

Solution:

Let $Q(t)$ denote the number of bacteria present in the culture at time *t*, where *t* is measured in hours. Then

$$\frac{dQ}{dt} = kQ$$

where *k* is a constant of proportionality.

Applied Example 1 – *Solution*

cont'd

Solving this separable first-order differential equation, we obtain

$$Q(t) = Q_0e^{kt} \qquad \text{Equation (8)}$$

where Q_0 denotes the initial bacteria population.

Since $Q_0 = 10{,}000$, we have

$$Q(t) = 10{,}000e^{kt}$$

Applied Example 1 – *Solution*

cont'd

Next, the condition that 60,000 bacteria are present 2 hours later translates into $Q(2) = 60,000$, or

$$60,000 = 10,000e^{2k}$$

$$e^{2k} = 6$$

$$e^{k} = 6^{1/2}$$

Thus, the number of bacteria present at any time t is given by

$$Q(t) = 10,000e^{kt}$$

$$= 10,000(e^{k})^{t}$$

Applied Example 1 – *Solution*

cont'd

$$= (10,000)6^{t/2}$$

In particular, the number of bacteria present in the culture at the end of 4 hours is given by

$$Q(4) = 10,000(6^{4/2})$$

$$= 360,000$$

Restricted Growth Models

Restricted Growth Models

We have seen that a differential equation describing a restricted growth model is given by

$$\frac{dQ}{dt} = k(C - Q) \qquad \textbf{(9)}$$

where both *k* and *C* are positive constants. To solve this separable first-order differential equation, we first separate the variables and then integrate, obtaining

$$\int \frac{dQ}{C - Q} = \int k\, dt$$

$$-\ln |C - Q| = kt + C_1$$

C_1, an arbitrary constant

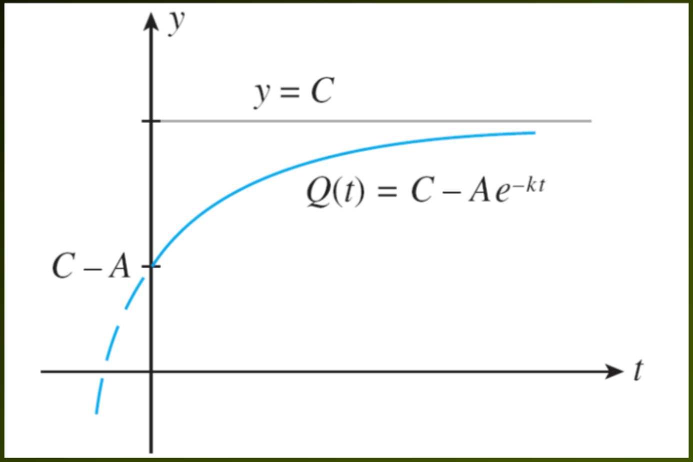
Restricted Growth Models

$$\ln |C - Q| = -kt - C_1 \qquad \textbf{(10)}$$

$$|C - Q| = e^{-kt - C_1} = e^{-kt}\, e^{-C_1} = C_2 e^{-kt} \qquad C_2 = e^{-C_1}$$

$$C - Q = Ae^{-kt} \qquad A = \pm C_2$$

This is the equation of the learning curve (Figure 8)



A restricted exponential growth model
Figure 8

Applied Example 2 – *Yield of a Wheat Field*

In an experiment conducted by researchers in the Agriculture Department of a midwestern university, it was found that the maximum yield of wheat in the university's experimental field station was 150 bushels per acre. Furthermore, the researchers discovered that the rate at which the yield of wheat increased was governed by the differential equation

$$\frac{dQ}{dx} = k(150 - Q)$$

where $Q(x)$ denotes the yield in bushels per acre and x is the amount in pounds of an experimental fertilizer used per acre of land.

Applied Example 2 – *Yield of a Wheat Field*

cont'd

Data obtained in the experiment indicated that 10 pounds of fertilizer per acre of land would result in a yield of 80 bushels of wheat per acre, whereas 20 pounds of fertilizer per acre of land would result in a yield of 120 bushels of wheat per acre. Determine the yield if 30 pounds of fertilizer were used per acre.

Solution:

The given differential equation has the same form as Equation (9) with $C = 150$. Solving it directly or using the result obtained in the solution of Equation (9), we see that the yield per acre is given by

$$Q(x) = 150 - Ae^{-kx}$$

Applied Example 2 – *Solution*

cont'd

The first of the given conditions implies that $Q(10) = 80$; that is,

$$150 - Ae^{-10k} = 80$$

or $$A = 70e^{10k}.$$

Therefore,

$$Q(x) = 150 - 70e^{10k}\,e^{-kx}$$

$$= 150 - 70e^{-k(x-10)}$$

Applied Example 2 – *Solution*

cont'd

The second of the given conditions implies that $Q(20) = 120$, or

$$150 - 70e^{-k(20-10)} = 120$$

$$70e^{-10k} = 30$$

$$e^{-10k} = \frac{3}{7}$$

Taking the logarithm of each side of the equation, we find

$$\ln e^{-10k} = \ln\left(\frac{3}{7}\right)$$

Applied Example 2 – *Solution*

cont'd

$$-10k = \ln 3 - \ln 7 \approx -0.8473$$

$$k \approx 0.085$$

Therefore,

$$Q(x) = 150 - 70e^{-0.085(x-10)}$$

In particular, when $x = 30$, we have

$$Q(30) = 150 - 70e^{-0.085(20)}$$

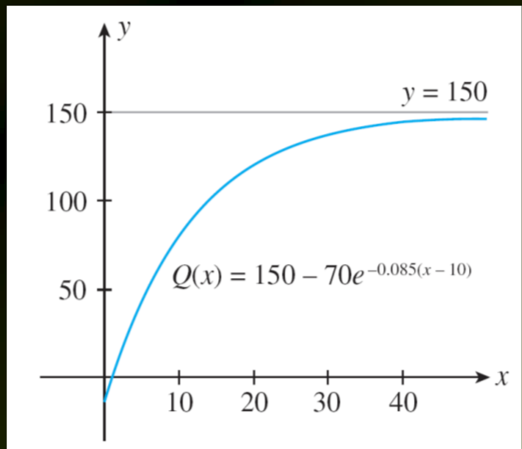
Applied Example 2 – *Solution*

cont'd

$$= 150 - 70e^{-1.7}$$

$$\approx 137$$

So the yield would be 137 bushels per acre if 30 pounds of fertilizer were used per acre. The graph of $Q(x)$ is shown in Figure 9.



$Q(x)$ is a function relating crop yield to the amount of fertilizer used.

Figure 9

Restricted Growth Models

Next, let's consider a differential equation describing another type of restricted growth:

$$\frac{dQ}{dt} = kQ(C - Q)$$

where *k* and *C* are positive constants. Separating variables and integrating each side of the resulting equation with respect to the appropriate variable, we have

$$\int \frac{1}{Q(C ? Q} dQ = \int k \; dt$$

As it stands, the integrand on the left side of this equation is not in a form that can be easily integrated.

Restricted Growth Models

However, observe that

$$\frac{1}{Q(C ? Q} = \frac{1}{C}\left[\frac{1}{Q} + \frac{1}{C ? Q}\right]$$

as you may verify by adding the terms between the brackets on the right-hand side. Making use of this identity, we have

$$\int \frac{1}{C}\left[\frac{1}{Q} + \frac{1}{C - Q}\right] dQ = \int k \; dt$$

$$\int \frac{dQ}{Q} + \int \frac{dQ}{C - Q} = Ck \int dt$$

$$\ln |Q| - \ln |C - Q| = Ckt + b$$

b, an arbitrary constant

Restricted Growth Models

$$\ln\left|\frac{Q}{C-Q}\right| = Ckt + b$$

$$\left|\frac{Q}{C-Q}\right| = e^{Ckt+b} = e^{b}\,e^{Ckt}$$

$$\frac{Q}{C-Q} = Be^{Ckt}$$

$B = \pm e^{b}$

$$Q = CBe^{Ckt} - QBe^{Ckt}$$

Restricted Growth Models

$$(1 + Be^{Ckt})Q = CBe^{Ckt}$$

and

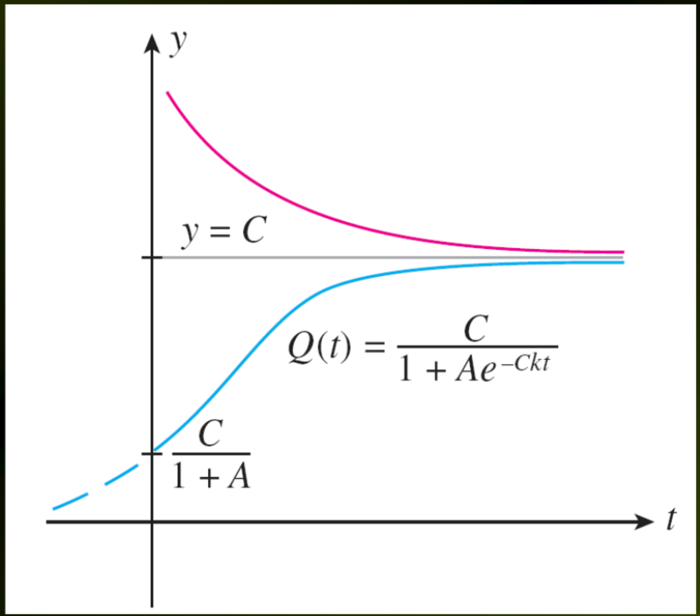
$$Q = \frac{CBe^{Ckt}}{1 + Be^{Ckt}}$$

or

$$Q(t) = \frac{C}{1 + Ae^{-Ckt}} \qquad A = \frac{1}{B} \qquad \textbf{(11)}$$

Restricted Growth Models

See Figure 2. In its final form, this function is equivalent to the logistic function.



Two solutions of the logistic equation

Figure 2