9 DIFFERENTIAL EQUATIONS

9.4

Approximate Solutions of Differential Equations

Euler’s Method

Euler's Method

As in the case of definite integrals, there are many differential equations whose exact solutions cannot be found using any of the available methods. In such cases, we must once again resort to approximate solutions.

Many numerical methods have been developed for efficient computation of approximate solutions to differential equations. In this section we will look at a method for solving the problem

$$\frac{dy}{dx} = F(x, y) \qquad y(x_0) = y_0 \qquad \textbf{(12)}$$

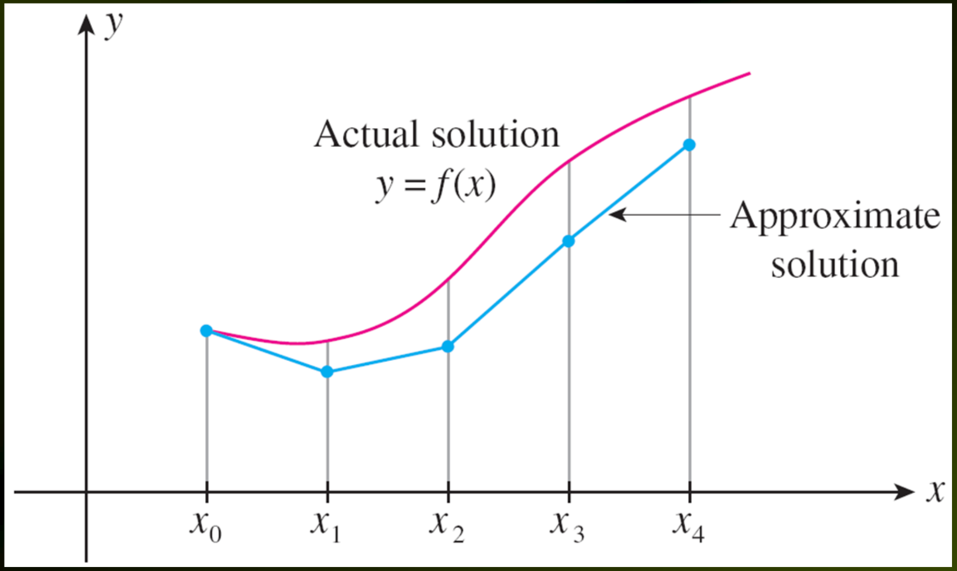
Euler's Method

Euler's method, named after Leonhard Euler, describes a way of finding an approximate solution of Equation (12). Basically, the technique calls for approximating the actual solution $y = f(x)$ at certain selected values of x.

The values of f between two adjacent values of x are then found by linear interpolation.

Euler’s Method

This situation is depicted geometrically in Figure 13. Thus, in Euler’s method, the actual solution curve of the differential equation is approximated by a suitable polygonal curve.



Using Euler’s method, the actual solution curve of the differential equation is approximated by a polygonal curve.

Figure 13

Euler's Method

To describe the method, let h be a small positive number and let $x_n = x_0 + nh$ ($n = 1, 2, 3, \ldots$); that is,

$$x_1 = x_0 + h \qquad x_2 = x_0 + 2h \qquad x_3 = x_0 + 3h \qquad \ldots$$

Thus, the points $x_0, x_1, x_2, x_3, \ldots$ are spaced evenly apart, and the distance between any two adjacent points is h units.

We begin by finding an approximation y_1 to the value of the actual solution, $f(x_1)$ at $x = x_1$. Observe that the *initial* condition $y(x_0) = y_0$ of (12) tells us that the point (x_0, y_0) lies on the solution curve.

Euler's Method

Euler's method calls for approximating the part of the graph of f on the interval $[x_0, x_1]$ by the straight-line segment that is tangent to the graph of f at (x_0, y_0). To find an equation of this straight-line segment, observe that the slope of this line segment is equal to $F(x_0, y_0)$.

So, using the point-slope form of an equation of a line, we see that the required equation is

$$y - y_0 = F(x_0, y_0)(x - x_0)$$

or

$$y = y_0 + F(x_0, y_0)(x - x_0)$$

Euler's Method

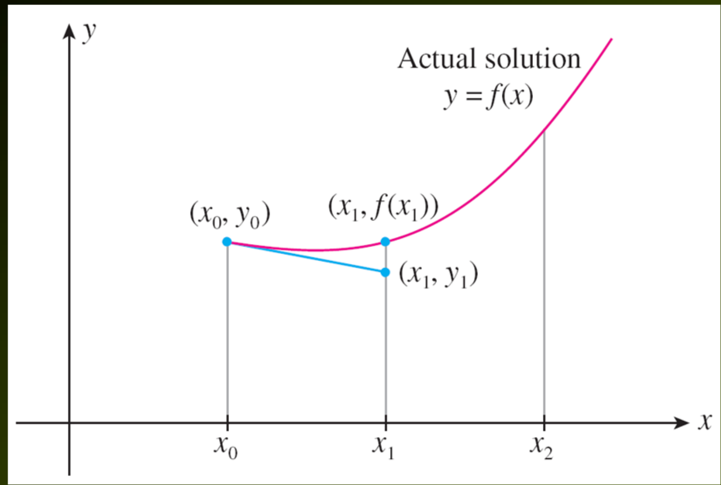
Therefore, the approximation y_1 to $f(x_1)$ is obtained by replacing x by x_1. Thus,

$$y_1 = y_0 + F(x_0, y_0)(x_1 - x_0)$$

$$= y_0 + F(x_0, y_0)h$$

Since $x_1 - x_0 = h$

This situation is depicted in Figure 14.



$y_1 = y_0 + hF(x_0, y_0)$ is the number used to approximate $f(x_1)$.

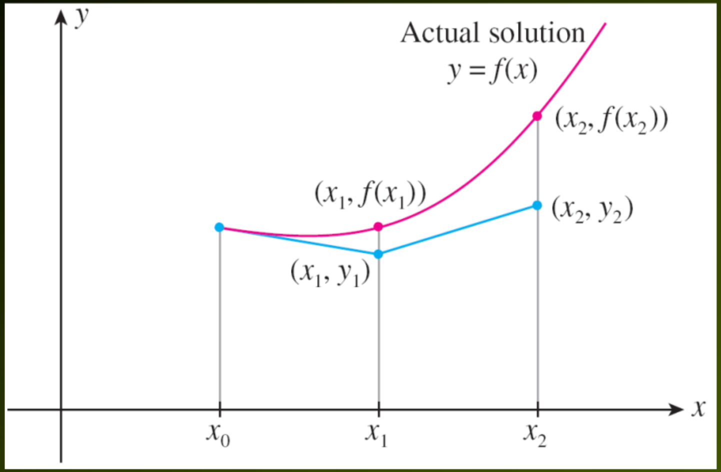
Figure 14

Euler's Method

Next, to find an approximation y_2 to the value of the actual solution, $f(x_2)$, at $x = x_2$, we repeat the preceding procedure but this time taking the slope of the straight-line segment on $[x_1, x_2]$ to be $F(x_1, y_1)$. We obtain

$$y_2 = y_1 + F(x_1, y_1)h$$

(See Figure 15.)



$y_2 = y_1 + hF(x_1, y_1)$ is the number used to approximate $f(x_2)$.

Figure 15

Euler's Method

Continuing in this manner, we see that $y_1, y_2, \ldots, y_n$ can be found by the general formula

$$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1}) \qquad (n = 1, 2, \ldots)$$

Euler's Method

We now summarize this procedure.

Euler's Method

Suppose we are given the differential equation

$$\frac{dy}{dx} = F(x, y)$$

subject to the initial condition $y(x_0) = y_0$ and we wish to find an approximation of $y(b)$, where b is a number greater than x_0 and n is a positive integer. Compute

$$h = \frac{b - x_0}{n}$$

$$x_1 = x_0 + h \qquad x_2 = x_0 + 2h \qquad x_3 = x_0 + 3h \qquad \ldots \qquad x_n = x_0 + nh = b$$

and

$$\begin{aligned} y_0 &= y(x_0) \\ y_1 &= y_0 + hF(x_0, y_0) \\ y_2 &= y_1 + hF(x_1, y_1) \\ &\vdots \\ y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1}) \end{aligned}$$

Then, y_n gives an approximation of the true value $y(b)$ of the solution to the initial value problem at $x = b$.

Solving Differential Equations with Euler's Method

Example 1

Use Euler's method with $n = 8$ to obtain an approximation of the solution of the initial value problem

$$y' = x - y \qquad y(0) = 1$$

when $x = 2$.

Solution:

Here, $x_0 = 0$ and $b = 2$, so taking $n = 8$, we find

$$h = \frac{2\,?}{8} = \frac{1}{4}$$

and

$$x_0 = 0 \qquad x_1 = \frac{1}{4} \qquad x_2 = \frac{1}{2} \qquad x_3 = \frac{3}{4} \qquad x_4 = 1$$

Example 1 – *Solution*

cont'd

$$x_5 = \frac{5}{4} \qquad x_6 = \frac{3}{2} \qquad x_7 = \frac{7}{4} \qquad x_8 = b = 2$$

Also,

$$F(x, y) = x - y$$

and

$$y_0 = y(0) = 1$$

Therefore, the approximations of the actual solution at the points $x_0, x_1, x_2, \ldots, x_n = b$ are

$$y_0 = y(0) = 1$$

Example 1 – *Solution*

cont'd

$$y_1 = y_0 + hF(x_0, y_0)$$

$$= 1 + \frac{1}{4}(0 \; ? \; 1)$$

$$= \frac{3}{4}$$

$$y_2 = y_1 + hF(x_1, y_1)$$

$$= \frac{3}{4} + \frac{1}{4}\left(\frac{1}{4} \; ? \; \frac{3}{4}\right)$$

$$= \frac{5}{8}$$

Example 1 – *Solution*

cont'd

$$y_3 = y_2 + hF(x_2, y_2)$$

$$= \frac{5}{8} + \frac{1}{4}\left(\frac{1}{2} ? \frac{5}{8}\right)$$

$$= \frac{19}{32}$$

$$y_4 = y_3 + hF(x_3, y_3)$$

$$= \frac{19}{32} + \frac{1}{4}\left(\frac{3}{4} ? \frac{19}{32}\right)$$

$$= \frac{81}{128}$$

Example 1 – *Solution*

cont'd

$$y_5 = y_4 + hF(x_4, y_4)$$

$$= \frac{81}{128} + \frac{1}{4}\left(1 ? \frac{81}{128}\right)$$

$$= \frac{371}{512}$$

$$y_6 = y_5 + hF(x_5, y_5)$$

$$= \frac{371}{512} + \frac{1}{4}\left(\frac{5}{4} ? \frac{371}{512}\right)$$

$$= \frac{1753}{2048}$$

Example 1 – *Solution*

cont'd

$$y_7 = y_6 + hF(x_6, y_6)$$

$$= \frac{1753}{2048} + \frac{1}{4}\left(\frac{3}{2} ? \frac{1753}{2048}\right)$$

$$= \frac{8331}{8192}$$

$$y_8 = y_7 + hF(x_7, y_7)$$

$$= \frac{8331}{8192} + \frac{1}{4}\left(\frac{7}{4} ? \frac{8331}{8192}\right)$$

$$= \frac{39{,}329}{32{,}768}$$

Example 1 – *Solution*

cont'd

Thus, the approximate value of $y(2)$ is

$$\frac{39{,}329}{32{,}768} \approx 1.2002$$